Take-home Exam 2

Question 1. [40 marks] Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{(1+(-1)^n)\sin\frac{n\pi}{4}\right\}$.

(b) $\left\{\frac{n+(-1)^n n^2}{n^2+1}\right\}$.

(c) $\{[1.5+(-1)^n]^n\}$.

(d) $\left\{\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}}\right\}$

Question 2. [40 marks] Let $\alpha > 0$. Choose $x_1 \geq \sqrt{\alpha}$. For $n = 1, 2, 3, \cdots$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Show that the sequence $\{x_n\}$ is bounded below by $\sqrt{\alpha}$ and monotone decreasing.

(b) Prove that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(c) Prove that $0 \leq x_n - \sqrt{\alpha} \leq \frac{x_n^2 - \alpha}{x_n}$.

(d) Let $\alpha = 3$ and $x_1 = 2$. Use part (c) to find x_n such that $\left|x_n - \sqrt{3}\right| < 10^{-8}$.

Hint: From the inequality $a^2 + b^2 \geq 2ab$, for $y > 0$,

$$y + \frac{\alpha}{y} \geq 2\sqrt{y}\cdot\sqrt{\frac{\alpha}{y}} = 2\sqrt{\alpha}.$$

Question 3. [20 marks] Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that

$$\underline{\lim}\, a_n + \overline{\lim}\, b_n \leq \overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$