

Take-home Exam 3

Question 1 [40 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

- (a) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1}$.
- (b) $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$.
- (c) $\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$.
- (d) $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.
- (e) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k}$.

Question 2. [20 marks]

- (a) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, prove that $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.
- (b) If a and b are positive real numbers, prove that

$$\sum_{k=1}^{\infty} \frac{1}{(ak + b)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Question 3. [20 marks] Determine all values of p and q for which the following series converges:

$$\sum_{k=2}^{\infty} \frac{1}{k^p (\ln k)^q}$$

(*Hint:* Consider the three cases $p > 1$, $p = 1$ and $p < 1$.)

Question 4. [20 marks] Suppose that $\sum_{k=1}^{\infty} a_k$ is a **convergent positive** series.

For each of the following, either prove that the given series converges, or provide an example for which the series diverges.

- (a) $\sum_{k=1}^{\infty} \frac{a_k}{1 + a_k}$.
- (b) $\sum_{k=1}^{\infty} \sqrt{a_k}$.

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$$(c) \sum_{k=1}^{\infty} \sqrt[k]{k} a_k.$$

$$(d) \sum_{k=1}^{\infty} \sqrt{\frac{a_k}{k}}.$$