2002/2003 Semester I MA2108 Advanced Calculus II

Take-home Exam 4

Question 1 [40 marks]
Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}+1}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{2k^2+(-1)^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kt}{k^2+3}, \qquad t \in \mathbb{R}$.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}\ln(\ln k)}{\sqrt{\ln k}+1}$.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}k^k}{(k+1)^k}$.

Question 2. [20 marks] Given that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12},$$

determine how large $n \in \mathbb{N}$ can be chosen so that $\left|\frac{\pi^2}{12} - S_n\right| < 10^{-6}$, where S_n is the n th partial sum of the series.

Question 3. [20 marks] Determine the domain of the two-variable function defined by

$$f(x,y) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{n^x}$$

converges.

(*Hint.* The question is to ask you to find all values of x and y such that the given series converges. You may consider the three cases: $x > 0$, $x = 0$ and $x < 0$.)

Question 4. [20 marks] Suppose that the sequence $\{b_n\}$ is monotone decreasing with $\lim_{n\to\infty} b_n = 0$. If $\{a_n\}$ is a sequence satisfying $|a_n| \le b_n - b_{n+1}$ for all n, prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.