Take-home Exam 5

Question 1 [20 marks]
For each of the following sequence of functions, determine whether is converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

(a) $\left\{\left(1-\frac{x^2}{n}\right)^{nx}\right\}, \quad x \in \mathbb{R}.$

(b) $\{(\cos x)^{2n}\}, \quad x \in \mathbb{R}.$

(c) $\left\{\frac{\sin nx}{\cos nx + nx}\right\}, \quad x \in [1, +\infty).$

(d) $\{f_n(x)\}, \quad f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}, \quad x \in \mathbb{R}.$

Question 2. [30 marks] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $F_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(b) $F_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, 1].$

(c) $F_n(x) = x + \frac{x}{n}\sin nx, \quad x \in [-a, a], \quad a > 0.$

(d) $F_n(x) = x + \frac{x}{n}\sin nx, \quad x \in \mathbb{R}.$

(e) $F_n(x) = \frac{x^n \sin nx}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(f) $F_n(x) = nx\left(1-x^2\right)^n, \quad x \in [0, 1].$

Question 3. [30 marks] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \quad x \in [0, \infty).$

(b) $\sum_{k=1}^{\infty} e^{-kx} x^k, \quad x \in [0, \infty).$

(c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}, \quad x \in [0, \infty).$

(d) $\sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \quad x \in [-1, 1].$

(e) $\sum_{n=0}^{\infty} \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2}\right), \quad x \in [0, 1].$

(f) $\sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k, \quad x \in (-2, 2).$

Question 4. [20 marks] Show that each of the following series converges uniformly on $[a, \infty)$ for any $a > 0$, but does not converge uniformly on $(0, +\infty)$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2x}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1+x}}$.