Take-home Exam 6

Question 1 [20 marks]
Find the radius of convergence of each of the following power series:

(a) $\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k$.

(b) $\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2} (x-1)^k$.

(c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k$

(d) $\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$.

(*Hint.* For part (d), you can use the root test to find the interval for which the series converges absolutely.)

Question 2. [20 marks] For each of the following, determine all values of x for which the given series converges.

(a) $\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k}$.

(b) $\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k$.

(c) $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k(1-x)^k}, \quad x \neq 1$.

(d) $\sum_{k=1}^{\infty} \frac{1}{kx^k}, \quad x \neq 0$.

Question 3. [20 marks] Using any applicable method, find the Taylor series of each of the following functions at the indicated point, and specify the interval on which the series converges to the function.

(a) $f(x) = \cos x^2, \quad x_0 = 0$.

(b) $f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad x_0 = 0$.

(c) $f(x) = \sqrt{x}, \quad x_0 = 1$.

(d) $f(x) = \frac{x^2}{1-x^2}, \quad x_0 = 0$.

Question 4. [20 marks] Using the power series expansion of $\frac{1}{1-x}$ and its derivatives, find

(a) $\sum_{k=1}^{\infty} kx^k, \quad |x| < 1$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} k^2 x^k, \quad |x| < 1$.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$.

Question 5. [20 marks]

(a) Use series to estimate the integral's value

$$\int_0^{0.1} \arctan x^2 \, dx$$

with an error of magnitude less than 10^{-8}.

(b) Let $f(x) = x^3 \sin x^9$. Find $f^{(30)}(0)$.

(c) Estimate $\sin \frac{1}{10}$ with an error of magnitude less than 10^{-8}.

(d) Estimate $\sqrt[3]{999}$ with an error of magnitude less than 10^{-8}.

(*Hint.* For part (d), first you may use binomial series to give the series expansion of $\sqrt[3]{999} = 10\sqrt[3]{1 - \frac{1}{10^3}}$, and next you need to use your own methods to determine an upper bound of the remainders. You can not apply the alternating series estimation to this question because the series expansion of $\sqrt[3]{999}$ is no longer alternating. You may read our example for computing $\frac{\pi}{6} = \arcsin \frac{1}{2}$ for how to handling an upper bound of remainders. *Another way* for determining an upper bound of the remainders is Taylor Theorem (Theorem 3.8.7). Of course you only need to use one of these methods to produce your solution.)