NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2001-2002

MA2108 Advanced Calculus II

April 2002 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\sqrt{\dfrac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}}\right\}$.

(b) $\left\{(\sqrt{n}+1)(\sqrt{3n+1} - \sqrt{3n})\right\}$.

(c) $\left\{\left(\dfrac{3n}{3n-1}\right)^{2n}\right\}$.

(d) $\left\{\dfrac{n^{50} \cdot 50^n \cdot \sin n}{n!}\right\}$.

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 2n - 1}{3n^3 + 2n^2 + n + 1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$.

(c) $\displaystyle\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.

Question 3 [12 marks]

(a) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{2n^2+\ln n}.$$

(b) Does the series of functions

$$\sum_{n=1}^{\infty} n^2 x^n \sin nx$$

converge uniformly on the interval $[0, \frac{1}{2}]$? Justify your answer.

Question 4 [16 marks]

(a) Find the general solutions of the following differential equations.

(i) $(1+t^2)y' + ty = 0.$

(ii) $\dfrac{dy}{dt} = \dfrac{y}{t} + 2\tan\left(\dfrac{y}{t}\right).$

(b) Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n+1}{\sqrt{n}}$ absolutely convergent, conditionally convergent or divergent? Justify your answer.

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\lim_{n\to\infty}\int_2^3 \frac{n^2\cdot\ln x\cdot\sin nx}{x^n}dx$. Justify your answer.

(b) Use an appropriate series to find a decimal approximation to $\sqrt{101}$ that is accurate to within 10^{-6}. Justify your answer.

(c) Let $\{a_n\}$ be a bounded sequence of real numbers and let f be a continuous function on $(-\infty,+\infty)$. Show that

$$f\left(\overline{\lim_{n\to\infty}}\, a_n\right)\le \overline{\lim_{n\to\infty}}\, f(a_n).$$

[Hint: Recall from tutorial that there exists a subsequence of $\{a_n\}$ convergent to $\overline{\lim_{n\to\infty}}\, a_n$.]

Question 6 [20 marks]

(a) Consider the function

$$f(x)=\sum_{n=1}^{\infty}\frac{xe^{-nx}}{n}.$$

Is $f(x)$ continuous on $(0,+\infty)$? Justify your answer.

(b) Consider the sequence $\{a_n\}$ defined recursively by

$$a_1=\sqrt{3},\quad a_{n+1}=\sqrt{3+a_n},\quad \text{for}\quad \text{n}\ge 1.$$

Show that $\{a_n\}$ converges, and find its limit.

(c) Let $\{a_n\}$ be a **convergent** sequence of real numbers. Suppose that the series $\sum_{n=1}^{\infty}\frac{a_n}{n}$ is **conditionally** convergent. Show that

(i) $\lim_{n\to\infty} a_n=0$.

(ii) $\overline{\lim_{n\to\infty}}\, n^p|a_n|=+\infty$ for any $p>0$.

Question 7 [20 marks]

(a) Evaluate the limit

$$\lim_{t\to 0} \frac{\sin(t^2) - \arctan(t^2)}{1 - \cos(t^3)}.$$

(b) Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of **positive** functions on an interval I and let $\{G_n(x)\}$ be a sequence of functions on I such that

$$\sup_{x\in I} |G_n(x)| < +\infty$$

for each $n \geq 1$. Suppose that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I, and the sequence of functions $\{G_n(x)\}$ converges uniformly to some function $G(x)$ on I.

(i) Show that

$$\sup_{x\in I} |G(x)| < +\infty.$$

(ii) Using (i) or otherwise, show that there exists a positive number M such that

$$|G_n(x)| \leq M$$

for all $x \in I$ and all $n \geq 1$.

(iii) Using (ii) or otherwise, show that the series of functions $\sum_{n=1}^{\infty} f_n(x)G_n(x)$ converges uniformly on I.