

**MA2108**

**Midterm**

**3 October 2003**

**Time allowed: 1.5 hours**

**Tutorial Group:**(circle one)

Wednesday 9-10

Wednesday 10-11

Wednesday 11-12

Wednesday 12-1

Thursday 12-1

Thursday 1-2

Thursday 2-3

Thursday 3-4

Friday 2-3

Friday 4-6

**Matriculation number:**\_\_\_\_\_

**Name:**\_\_\_\_\_

**Signature:**\_\_\_\_\_

Problem #	Your Grades
1 (10 points)	
2 (10 points)	
3 (10 points)	
4 (10 points)	
5 (10 points)	
6 (10 points)	
7 (10 points)	
8 (10 points)	
9 (10 points)	
10 (10 points)	
total (100 points)	

## MA2108 Midterm

3 October

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Determine the limit of the following sequences:

$$1. \left\{ (-1)^n \sin\left(\frac{1}{n}\right) + \arctan\left(\frac{2n^2}{4 + 3n + 2n^2}\right) \right\}.$$

*Solution.*

$$\begin{aligned} & \lim_{n \rightarrow \infty} (-1)^n \sin\left(\frac{1}{n}\right) + \arctan\left(\frac{2n^2}{4 + 3n + 2n^2}\right) \\ &= \lim_{n \rightarrow \infty} (-1)^n \sin\left(\frac{1}{n}\right) + \arctan\left(\frac{2}{4/n^2 + 3/n + 2}\right) = 0 + \arctan 1 = \frac{\pi}{4}, \end{aligned}$$

where  $\lim_{n \rightarrow \infty} (-1)^n \sin\left(\frac{1}{n}\right) = 0$  because

$$-\sin\left(\frac{1}{n}\right) \leq (-1)^n \sin\left(\frac{1}{n}\right) \leq \sin\left(\frac{1}{n}\right)$$

and

$$\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0.$$

□

$$2. \left\{ \frac{3^n + n^{60}60^n + 2n!}{n! + (\ln n)^2} \right\}.$$

*Solution.*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{3^n + n^{60}60^n + 2n!}{n! + (\ln n)^2} \\ &= \frac{3^n/n! + n^{60}/2^n \cdot 120^n/n! + 2}{1 + (\ln n)^2/n^2 \cdot n^2/2^n \cdot 2^n/n!} = \frac{0 + 0 + 2}{1 + 0 \cdot 0 \cdot 0} = 2. \end{aligned}$$

□

3.  $\left\{ \frac{\ln(2 + e^n)}{4n + 1} \right\}$ .

*Solution.*

$$\lim_{n \rightarrow \infty} \frac{\ln(2 + e^n)}{4n + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2+e^n} \cdot e^n}{4} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2/e^n + 1}}{4} = \frac{1}{4}.$$

□

*Another Solution.*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln(2 + e^n)}{4n + 1} &= \lim_{n \rightarrow \infty} \frac{\ln e^n (2/e^n + 1)}{4n + 1} = \lim_{n \rightarrow \infty} \frac{n + \ln(2/e^n + 1)}{4n + 1} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \frac{\ln(2/e^n + 1)}{n}}{4 + 1/n} = \frac{1 + 0}{4 + 0} = \frac{1}{4}. \end{aligned}$$

□

4.  $\left\{ (1 + 2n)^{1/n} \right\}$ .

*Solution.*

$$\lim_{n \rightarrow \infty} (1 + 2n)^{1/n} = \lim_{n \rightarrow \infty} n^{1/n} \cdot (1/n + 2)^{1/n} = 1 \cdot 1 \cdot (0 + 2)^0 = 1.$$

□

*Another Solution.*

$$\begin{aligned} \lim_{n \rightarrow \infty} (1 + 2n)^{1/n} &= \lim_{n \rightarrow \infty} \left( (1 + 2n)^{1/2n+1} \right)^{(2n+1)/n} \\ &= \lim_{n \rightarrow \infty} \left( (1 + 2n)^{1/2n+1} \right)^{2+1/n} = 1^2 = 1. \end{aligned}$$

□

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Determine convergence or divergence of the following series:

$$5. \sum_{n=1}^{\infty} \frac{2 + \sqrt{n}}{n^2 + 4n + 3}.$$

*Solution.* Let  $a_n = \frac{2 + \sqrt{n}}{n^2 + 4n + 3}$  and let  $b_n = \frac{1}{n^{3/2}}$ . Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2 + \sqrt{n}}{n^2 + 4n + 3} \cdot n^{3/2} = \lim_{n \rightarrow \infty} \frac{2/\sqrt{n} + 1}{1 + 4/n + 3/n^2} = 1.$$

Since  $\sum_{n=1}^{\infty} b_n$  converges by the  $p$ -series, the series  $\sum_{n=1}^{\infty} a_n$  converges. □

$$6. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}.$$

*Solution.* Let  $a_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$  and let  $b_n = \frac{1}{n^{3/2}}$ . Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n(n+1)(n+2)}} \cdot n^{3/2} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 \cdot (1 + 1/n) \cdot (1 + 2/n)}} = 1.$$

Since  $\sum_{n=1}^{\infty} b_n$  converges by the  $p$ -series, the series  $\sum_{n=1}^{\infty} a_n$  converges. □

Determine the absolute convergence, conditional convergence or divergence of the following series:

$$7. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}.$$

*Solution.* Let  $b_n = \frac{n}{n^2+1}$  and let  $f(x) = \frac{x}{x^2+1}$ . Since

$$f'(x) = \frac{(x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0$$

for  $x \geq 1$ , the sequence  $b_n$  is monotone decreasing. Since in addition

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n + 1/n} = 0,$$

the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$  converges by the alternating series test. Consider the series

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^2 + 1} \right| = \sum_{n=1}^{\infty} b_n$$

and let  $a_n = \frac{1}{n}$ . Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^2} = \lim_{n \rightarrow \infty} 1 + 1/n^2 = 1$$

and  $\sum_{n=1}^{\infty} a_n$  diverges by the  $p$ -series, the series  $\sum_{n=1}^{\infty} b_n$  diverges and so the series in the question converges conditionally.  $\square$

$$8. \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{n}{n+1} \right)^{n^2}.$$

*Solution.* Let  $a_n = (-1)^{n+1} \left( \frac{n}{n+1} \right)^{n^2}$ . Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \left| (-1)^{n+1} \left( \frac{n}{n+1} \right)^{n^2} \right|^{1/n} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1, \end{aligned}$$

the series converges absolutely by the root test for general series.  $\square$

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9. Find limit inferior and limit superior of the sequences:

$$\left\{ \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} \right\}$$

*Solution.*

$$\left\{ \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} \right\} = \left\{ \frac{2}{\sqrt{2}}, 1, \frac{2}{\sqrt{2}}, 0, -\frac{2}{\sqrt{2}}, 1, -\frac{2}{\sqrt{2}}, 0, \frac{2}{\sqrt{2}}, 1, \frac{2}{\sqrt{2}}, 0, -\frac{2}{\sqrt{2}}, 1, -\frac{2}{\sqrt{2}}, 0, \dots \right\}$$

The subsequential limits are  $-\frac{2}{\sqrt{2}} = -\sqrt{2}$ ,  $0$ ,  $1$ ,  $\frac{2}{\sqrt{2}} = \sqrt{2}$ . Thus

$$\overline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = \sqrt{2}$$

$$\underline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = -\sqrt{2}.$$

□

*Solution 2.* Let  $a_n = \left( \sin \frac{n\pi}{4} \right)^{(-1)^n}$ . Then

$$a_n = \begin{cases} 0 & n = 8k \\ \frac{2}{\sqrt{2}} & n = 8k + 1 \\ 1 & n = 8k + 2 \\ \frac{2}{\sqrt{2}} & n = 8k + 3 \\ 0 & n = 8k + 4 \\ -\frac{2}{\sqrt{2}} & n = 8k + 5 \\ 1 & n = 8k + 6 \\ -\frac{2}{\sqrt{2}} & n = 8k + 7 \end{cases}$$

The subsequential limits are  $-\frac{2}{\sqrt{2}} = -\sqrt{2}$ ,  $0$ ,  $1$ ,  $\frac{2}{\sqrt{2}} = \sqrt{2}$ . Thus

$$\overline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = \sqrt{2}$$

$$\underline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = -\sqrt{2}.$$

□

*Solution 3.* Let  $a_n = \left( \sin \frac{n\pi}{4} \right)^{(-1)^n}$ . Then

$$-\sqrt{2} \leq a_n \leq \sqrt{2}$$

and so

$$-\sqrt{2} \leq \underline{\lim}_{n \rightarrow \infty} a_n \leq \overline{\lim}_{n \rightarrow \infty} a_n \leq \sqrt{2}.$$

Since

$$\lim_{k \rightarrow \infty} a_{8k+1} = \sqrt{2} \quad \lim_{k \rightarrow \infty} a_{8k+5} = -\sqrt{2},$$

$$\overline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = \sqrt{2}$$

$$\underline{\lim}_{n \rightarrow \infty} \left( \sin \frac{n\pi}{4} \right)^{(-1)^n} = -\sqrt{2}.$$

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10. Show that the sequence defined by

$$a_1 = 2 \quad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies  $0 < a_n \leq 2$  and is monotone decreasing. Deduce that the sequence is convergent and find its limit.

*Solution.* We prove that  $0 < a_n \leq 2$  by induction. When  $n = 1$ ,  $0 < a_1 = 2 \leq 2$ . Suppose that  $0 < a_k \leq 2$ . Then  $1 \leq 3 - a_k < 3$  and so

$$0 < a_{k+1} = \frac{1}{3 - a_k} \leq 1 \leq 2.$$

The induction is finished and so  $0 < a_n \leq 2$  for all  $n$ .

We prove that  $a_{n+1} - a_n \leq 0$  by induction. When  $n = 1$ ,

$$a_2 = \frac{1}{3 - a_1} = 1 \leq a_1.$$

Suppose that  $a_{k+1} - a_k \leq 0$ . Then

$$\begin{aligned} a_{k+2} - a_{k+1} &= \frac{1}{3 - a_{k+1}} - \frac{1}{3 - a_k} = \frac{(3 - a_k) - (3 - a_{k+1})}{(3 - a_{k+1})(3 - a_k)} \\ &= \frac{a_{k+1} - a_k}{(3 - a_{k+1})(3 - a_k)} \leq 0 \end{aligned}$$

because  $a_{k+1} - a_k \leq 0$  by induction, and  $3 - a_{k+1} > 0, 3 - a_k > 0$  from the above. The induction is finished and so  $\{a_n\}$  is monotone decreasing.

Let  $A = \lim_{n \rightarrow \infty} a_n$ . Then

$$A = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{3 - a_n} = \frac{1}{3 - A}$$

and so  $3A - A^2 = 1$  or  $A^2 - 3A + 1 = 0$ . It follows that

$$A = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Since  $0 < a_n \leq 2$ ,  $0 \leq A \leq 2$  and the root  $A = \frac{3 + \sqrt{5}}{2}$  is rejected. Hence  $A = \frac{3 - \sqrt{5}}{2}$ , that is,

$$\lim_{n \rightarrow \infty} a_n = \frac{3 - \sqrt{5}}{2}.$$