NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA2108 ADVANCED CALCULUS II

November 2003 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\arctan\left(\sin\dfrac{(\ln n)^2}{\sqrt{n}}\right)+3+\dfrac{n^5}{1.1^n}\right\}$.

(b) $\left\{\dfrac{(\ln n)^3+8n!+6^n\cdot n^{100}}{n!+n^9}\right\}$.

(c) $\left\{\left(1+2n^2\right)^{1/n}\right\}$.

(d) $\left\{\sqrt[n]{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}}\right\}$.

Solution. (a)

$$\lim_{n\to\infty}\arctan\left(\sin\frac{(\ln n)^2}{\sqrt{n}}\right)+3+\frac{n^5}{1.1^n}=\arctan(\sin 0)+3+0=3.$$

(b)

$$\lim_{n\to\infty}\frac{(\ln n)^3+8n!+6^n\cdot n^{100}}{n!+n^9}=\lim_{n\to\infty}\frac{(\ln n)^3/n!+8+12^n/n!\cdot n^{100}/2^n}{1+n^9/n!}$$
$$=\frac{0+8+0\cdot 0}{1+0}=8$$

(c)

$$\lim_{n\to\infty}\left(1+2n^2\right)^{1/n}=\lim_{n\to\infty}\left(n^{1/n}\right)^2\left(1/n^2+2\right)^{1/n}=1^2\cdot 2^0=1$$

(d) Let $a_n=\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}$. Then

$$\lim_{n\to\infty}\sqrt[n]{\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}}=\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$
$$=\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{(n+1)!}\cdot\frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$$
$$=\lim_{n\to\infty}\frac{2n+1}{n+1}=\lim_{n\to\infty}\frac{2+1/n}{1+1/n}=2$$

∎

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $$\sum_{n=1}^{\infty} \frac{n^2 - 6n}{n^3 + 3n + 1}.$$

(b) $$\sum_{n=1}^{\infty} \left(\frac{5n^2 + 1}{1 + 8n^2}\right)^n.$$

(c) $$\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}.$$

(d) $$\sum_{n=1}^{\infty} \sin\left(\frac{n}{1+n^2}\right).$$

Solution. (a). Divergence. Let $a_n = \frac{n^2-6n}{n^3+3n+1}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 - 6n}{n^3 + 3n + 1} \cdot n = \lim_{n\to\infty} \frac{1 - 6/n}{1 + 3/n^2 + 1/n^3} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, $\sum_{n=1}^{\infty} a_n$ divergence by the limit comparison test.

(b). Convergence by the root test because

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{5n^2+1}{1+8n^2}\right)^n} = \lim_{n\to\infty} \frac{5n^2+1}{1+8n^2} = \lim_{n\to\infty} \frac{5 + 1/n^2}{1/n^2 + 8} = \frac{5}{8} < 1.$$

(c). Convergence by the ratio test because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)(3n+5)} \cdot \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{3n+5} = \lim_{n\to\infty} \frac{1 + 1/n}{3 + 5/n} = \frac{1}{3} < 1.$$

(d) Divergence. Let $a_n = \sin\left(\frac{n}{1+n^2}\right)$ and let $b_n = \frac{1}{n}$. Then $a_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{n}{1+n^2}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\left(\frac{n}{1+n^2}\right)}{\frac{n}{1+n^2}} \cdot \frac{n^2}{1+n^2}$$

$$\overset{x=\frac{n}{1+n^2}}{=\!=\!=} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{n\to\infty} \frac{1}{1/n^2 + 1} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, $\sum_{n=1}^{\infty} a_n$ diverges by the limit comparison test.

∎

Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n2^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$.

Solution. (a).

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n2^n} = \sum_{k=1}^{\infty} \frac{1}{n} \cdot \left(x - \frac{1}{2}\right)^n$$

The radius of convergence

$$R = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{n}}} = \frac{1}{\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}} = 1.$$

(b) The radius of convergence

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim_{n\to\infty} \frac{[(n+1)!]^3}{(3n+3)!} \frac{(3n)!}{(n!)^3}}$$

$$= \frac{1}{\lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}} = \frac{1}{\lim_{n\to\infty} \frac{(1+1/n)^3}{(3+3/n)(3+2/n)(3+1/n)}} = 27$$

∎

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converges uniformly on the indicated intervals. Justify your answers.

$$F_n(x) = 2x(1-x)^n\sqrt{n}, \quad x \in [0,1].$$

(b) Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{\sqrt{k^3+x^3}}, \quad x \in [0,\infty).$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+2x}} \quad x \in [0,\infty).$

Solution. (a) When $0 < x < 1$, then

$$\lim_{n\to\infty} 2x(1-x)^n\sqrt{n} = \lim_{n\to\infty} 2x\frac{\sqrt{n}}{(1/(1-x))^n} = 0$$

because $1/(1-x) > 1$ and $F_n(0) = F_n(1) = 0$. Thus the limiting function $F(x) = \lim_{n\to\infty} F_n(x) = 0$ for $x \in [0,1]$. Since $F_n(x) \geq 0$ for $x \in [0,1]$,

$$T_n = \sup_{0\leq x\leq 1} |F_n(x) - F(x)| = \sup_{0\leq x\leq 1} F_n(x).$$

From

$$F_n'(x) = 2\sqrt{n}(1-x)^n + 2\sqrt{n}x\cdot n(1-x)^{n-1}\cdot(-1) = 2\sqrt{n}(1-x)^{n-1}(1-(n+1)x) = 0,$$

$x = \frac{1}{n+1}$ or 1 and so

$$T_n = \max_{0\leq x\leq 1} F_n(x) = \max\left\{F_n(0), F_n\left(\frac{1}{n+1}\right), F_n(1)\right\} = 2\sqrt{n}\cdot\frac{1}{n+1}\cdot\left(1-\frac{1}{n+1}\right)^n.$$

Since

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \frac{2}{\sqrt{n}+\frac{1}{\sqrt{n}}} \cdot \left[\left(1-\frac{1}{n+1}\right)^{n+1}\right]^{1/(1+1/n)} = 0$$

the sequence of functions converges uniformly on $[0,1]$ by the T-test.

(b) (i). Let $f_k(x) = \frac{\cos kx}{\sqrt{k^3+x^3}}$. Then

$$|f_k(x)| = \left|\frac{\cos kx}{\sqrt{k^3+x^3}}\right| \leq \frac{1}{k^{3/2}}.$$

Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ converges by p-series, the series of functions $\displaystyle\sum_{n=1}^{\infty} f_k(x)$ converges uniformly by Weierstrass M-test.

(b)(ii). Let $b_k = \frac{1}{\sqrt{k+2x}}$. Then for $x \geq 0$, b_k is positive and monotone decreasing with $\lim_{k\to\infty} b_k = 0$. By the alternating series test, the series of functions $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k+2x}}$ converges pointwise on $[0,\infty)$. By alternating series estimation,

$$0 \leq T_n = \sup_{x \geq 0} \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{\sqrt{k+2x}} \right| \leq \sup_{x \geq 0} \frac{1}{\sqrt{n+1+2x}} \leq \frac{1}{\sqrt{2n+1}}.$$

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}} = \lim\limits_{n\to\infty} 0$, $\lim\limits_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the series of functions converges uniformly on $[0, \infty)$.

∎

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\lim_{n\to\infty} \int_0^1 \left(\frac{x^2+x+2}{5}\right)^n \cos(nx^2)\ dx$. Justify your answer.

(b) Find the **interval of convergence** of the power series

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{1+n\ln n}.$$

Justify your answer.

(c) Let $\{a_n\}$ be a **convergent** sequence. Define

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Does the sequence $\{b_n\}$ converge? Justify your answer.

Solution. (a) Let $F_n(x) = \left(\frac{x^2+x+2}{5}\right)^n \cos(nx^2)$. Then, for $0 \le x \le 1$,

$$-\left(\frac{4}{5}\right)^n \le F_n(x) \le \left(\frac{4}{5}\right)^n.$$

Since $\lim_{n\to\infty} \left(\frac{4}{5}\right)^n = 0$,

$$F(x) = \lim_{n\to\infty} F_n(x) = 0$$

by the Squeeze Theorem for $0 \le x \le 1$. Now

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left(\frac{x^2+x+2}{5}\right)^n |\cos(nx^2)| \le \left(\frac{4}{5}\right)^n.$$

Since $\lim_{n\to\infty} \left(\frac{4}{5}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so

$$\lim_{n\to\infty} \int_0^1 \left(\frac{x^2+x+2}{5}\right)^n \cos(nx^2)\ dx$$

$$= \int_0^1 \lim_{n\to\infty} \left(\frac{x^2+x+2}{5}\right)^n \cos(nx^2)\ dx = \int_0^1 0\ dx = 0.$$

(b) The power series

$$\sum_{n=1}^{\infty}\frac{(2x+1)^n}{1+n\ln n}=\sum_{n=1}^{\infty}\frac{2^n}{1+n\ln n}\left(x+\frac{1}{2}\right)^n$$

has the radius of convergence

$$R=\frac{1}{\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}}=\frac{1}{\lim\limits_{n\to\infty}\frac{2^{n+1}}{1+(n+1)\ln(n+1)}\cdot\frac{1+n\ln n}{2^n}}$$

$$=\frac{1}{\lim\limits_{n\to\infty}\frac{2}{\frac{1}{n\ln n}+\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln n}}\cdot\left(\frac{1}{n\ln n}+1\right)}=\frac{1}{2}$$

with $x_0=-\frac{1}{2}$. When $x=-\frac{1}{2}-\frac{1}{2}=-1$, the series

$$\sum_{n=1}^{\infty}\frac{(2x+1)^n}{1+n\ln n}=\frac{(-1)^n}{1+n\ln n}$$

converges by the alternating series test because $\frac{1}{1+n\ln}$ is positive and monotone decreasing with $\lim\limits_{n\to\infty}\frac{1}{1+n\ln n}=0$. When $x=-\frac{1}{2}+\frac{1}{2}=0$, the series

$$\sum_{n=1}^{\infty}\frac{(2x+1)^n}{1+n\ln n}=\frac{1}{1+n\ln n}$$

diverges by limit comparison test with $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ because

$$\lim_{n\to\infty}\frac{\frac{1}{1+n\ln n}}{\frac{1}{n\ln n}}=1$$

and the series $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges by the integral test because, let $f(x)=\frac{1}{x\ln x}$, then $f(x)$ is positive monotone decreasing with

$$\int_2^{\infty}f(x)\,dx=\int_2^{\infty}\frac{1}{x\ln x}\,dx\overset{u=\ln x}{=\!=\!=}\int_{\ln 2}^{\infty}\frac{du}{u}=+\infty.$$

Hence the interval of convergence is $[-1,0)$.

(c) Since $\{a_n\}$ converges, $\{a_n\}$ is bounded and so there exists a positive number M such that $|a_n|\le M$ for all n. Let $A=\lim\limits_{n\to\infty}a_n$. Given any $\epsilon>0$, there exists N_1 such that

$$|a_n-A|<\frac{\epsilon}{2}$$

for $n > N_1$. Let N be the smallest integer such that $N > \max\{\frac{2N_1(M+|A|)}{\epsilon}, N_1\}$. For $n > N$,

$$|b_n - A| = \left|\frac{(a_1 - A) + (a_2 - A) + \cdots + (a_n - A)}{n}\right|$$

$$\leq \left|\frac{(a_1 - A) + \cdots + (a_{N_1} - A)}{n}\right| + \left|\frac{(a_{N_1+1} - A) + \cdots + (a_n - A)}{n}\right|$$

$$\leq \frac{|a_1| + |A| + \cdots + |a_{N_1}| + |A|}{n} + \frac{|a_{N_1+1} - A| + \cdots + |a_n - A|}{n}$$

$$\leq \frac{N_1(M + |A|)}{n} + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and hence $\lim_{n\to\infty} b_n = A$. ∎

Question 6 [20 marks]

(a) Using any applicable method, find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt[4]{16 - 2x}}$ at $x_0 = 0$, and determine its radius of convergence.

(b) Suppose that the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges on an interval I. Prove that the sequence of functions $\{f_n(x)\}$ converges uniformly to 0 on I.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Show that

$$\overline{\lim_{n\to\infty}}\,(a_n + b_n) \leq \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

Solution. (a) By using binomial series,

$$f(x) = (16 - 2x)^{-\frac{1}{4}} = 1/2\left(1 - \frac{x}{8}\right)^{-\frac{1}{4}} = 1/2\left(1 + \sum_{n=1}^{\infty}\binom{-\frac{1}{4}}{n}\left(\frac{x}{8}\right)^n\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\binom{-\frac{1}{4}}{n}\frac{x^n}{2\cdot 8^n}$$

for

$$\left|-\frac{x}{8}\right| < 1.$$

Thus radius of convergence is 8.

(b). Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Since $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I, the sequence of partial sums $\{S_n(x)\}$ converges uniformly on I. By Cauchy Criterion, given any $\epsilon > 0$, there exists N such that

$$|S_m(x) - S_n(x)| < \epsilon$$

for all $x \in I$ and $n, m > N$. In particular,

$$|f_n(x) - 0| = |S_n(x) - S_{n-1}(x)| < \epsilon$$

for $n > N+1$ and hence $\{f_n(x)\}$ converges uniformly to 0 by definition.

(c) For each $m \geq n$, we have

$$a_m + b_m \leq \sup\{a_k \mid k \geq n\} + \sup\{b_k \mid k \geq n\}.$$

Thus $\sup\{a_k \mid k \geq n\} + \sup\{b_k \mid k \geq n\}$ is an upper bound of the set $\{a_m + b_m \mid m \geq n\} = \{a_k + b_k \mid k \geq n\}$. It follows that

$$\sup\{a_k + b_k \mid k \geq n\} \leq \sup\{a_k \mid k \geq n\} + \sup\{b_k \mid k \geq n\}$$

because sup is the least upper bound, and so, by letting n tend to infinity,

$$\begin{aligned} \overline{\lim_{n\to\infty}} (a_n + b_n) &= \lim_{n\to\infty} (\sup\{a_k + b_k \mid k \geq n\}) \\ &\leq \lim_{n\to\infty} (\sup\{a_k \mid k \geq n\} + \sup\{b_k \mid k \geq n\}) \end{aligned}$$

$$= \lim_{n\to\infty} (\sup\{a_k \mid k \geq n\}) + \lim_{n\to\infty} (\sup\{b_k \mid k \geq n\}) = \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

∎

Question 7 [20 marks]

(a) Let $f(x) = x^3 \cdot \sin(x^8)$. Find $f^{(48)}(0)$.

(b) Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1 \qquad a_{n+1} = 1 + \frac{a_n}{a_n + 1} \quad n \geq 1.$$

Prove that $\{a_n\}$ is convergent, and find its limit.

(c) Show that the series

$$1 + \sum_{n=1}^{\infty} \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

converges for any real number $a > -1$.

Solution. (a) By using Taylor series,

$$f(x) = x^3 \cdot \sin(x^8) = x^3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left(x^8\right)^{2k-1}}{(2k-1)!} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{8(2k-1)+3}}{(2k-1)!}.$$

Compare with

$$f(x) = f(0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}.$$

By the uniqueness theorem of power series,

$$f^{(48)}(0) = 0.$$

(b) First we prove that $1 \leq a_n \leq 2$ by induction. This is true for $n = 1$. Assume that $1 \leq a_{n-1} \leq 2$. Then

$$1 \leq 1 + \frac{a_n}{a_n + 1} = a_{n+1} \leq 2.$$

The induction is finished and so $1 \leq a_n \leq 2$ for all n.

Next we prove that $\{a_n\}$ is monotone increasing. Note that $a_2 = \frac{3}{2} > a_1 = 1$. Assume that $a_n \geq a_{n-1}$. Then

$$a_{n+1} - a_n = \left(1 + \frac{a_n}{a_n + 1}\right) - \left(1 + \frac{a_{n-1}}{a_{n-1} + 1}\right) = \frac{a_n - a_{n-1}}{(a_n + 1)(a_{n-1} + 1)} \geq 0.$$

By monotone convergence theorem, $\{a_n\}$ converges. Let $A = \lim_{n\to\infty} a_n$. Then

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(1 + \frac{a_n}{a_n + 1}\right) = 1 + \frac{A}{A+1}$$

and so $A^2 - A - 1 = 0$. It follows that

$$A = \frac{1 \pm \sqrt{5}}{2}.$$

Since $1 \le a_n \le 2$, $A = \frac{1-\sqrt{5}}{2}$ is rejected and hence $A = \frac{1+\sqrt{5}}{2}$, that is, $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.

(c) Observe that this series is eventually alternating. Let

$$a_n = \binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}.$$

Then

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{a(a-1)\cdots(a-n+1)(a-n)}{(n+1)!} \cdot \frac{n!}{a(a-1)(a-2)\cdots(a-n+1)} \right|$$

$$= \frac{|a-n|}{n+1} = \frac{n-a}{n+1} \le 1$$

because $a > -1$. Thus $\{|a_n|\}$ is monotone decreasing. From

$$|a_n| = \left|1 - \frac{a+1}{1}\right| \cdot \left|1 - \frac{a+1}{2}\right| \cdot \cdots \cdot \left|1 - \frac{a+1}{n}\right|,$$

$$-\ln|a_n| = \sum_{k=1}^{n} -\ln\left|1 - \frac{a+1}{k}\right|.$$

Since $-\ln\left|1 - \frac{a+1}{k}\right| \ge 0$ for $k \ge a+1$, the series

$$\sum_{k=1}^{\infty} -\ln\left|1 - \frac{a+1}{k}\right|$$

is eventually positive. Since

$$\lim_{k\to\infty} \frac{-\ln\left|1 - \frac{a+1}{k}\right|}{\frac{1}{k}} = \lim_{k\to\infty} \frac{-\ln\left(1 - \frac{a+1}{k}\right)}{\frac{1}{k}}$$

$$\overset{x=\frac{1}{k}}{=\!=\!=} = \lim_{x\to 0} \frac{-\ln(1-(a+1)x)}{x} = \lim_{x\to 0} -\frac{1}{1-(a+1)x} \cdot (-a-1) = a+1 > 0$$

(because $a > -1$) and the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the (eventually positive) series

$$\sum_{k=1}^{\infty} -\ln\left|1 - \frac{a+1}{k}\right|$$

diverges by limit comparison test and so

$$\lim_{n\to\infty} -\ln|a_n| = \lim_{n\to\infty} \sum_{k=1}^{n} -\ln\left|1 - \frac{a+1}{k}\right| = +\infty$$

or $\lim_{n\to\infty} \ln|a_n| = -\infty$. It follows that

$$\lim_{n\to\infty} |a_n| = 0.$$

By alternating series test, the series $1 + \sum_{n=1}^{\infty} \binom{a}{n}$ converges. ∎