

Solutions to Take-home Exam 1

Question 1. [2 points, 1 for each part]

Prove the following limits by using $\epsilon - N$ definition

- i) $\lim_{n \rightarrow \infty} \frac{3n+8}{2n+9} = \frac{3}{2}$.
 ii) $\lim_{n \rightarrow \infty} \frac{(-1)^n n}{n^2+1} = 0$.

Proof. (i). Note that

$$\left| \frac{3n+8}{2n+9} - \frac{3}{2} \right| = \left| \frac{2(3n+8) - 3(2n+9)}{2(2n+9)} \right| = \frac{11}{2(2n+9)} < \frac{11}{4n} < \frac{3}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{3}{N} \leq \epsilon \iff N \geq \frac{3}{\epsilon}$. When $n > N$,

$$\left| \frac{3n+8}{2n+9} - \frac{3}{2} \right| < \frac{3}{n} < \frac{3}{N} \leq \epsilon.$$

(ii). Note that

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{1}{N} \leq \epsilon \iff N \geq \frac{1}{\epsilon}$. When $n > N$,

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

□

Question 2. [5 points, 1 for each part]

For each of the following sequences, either find the limit or show that the limit does not exist.

- (a) $\left\{ \left(\sqrt{n^2+n} - n \right) \right\}$.
 (b) $\left\{ (2^n + 3^n)^{\frac{1}{n}} \right\}$.
 (c) $\left\{ \sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}} \right\}$.
 (d) $\left\{ \left(\frac{3n}{3n-1} \right)^{2n+\sqrt{n}} \right\}$.
 (e) $\left\{ \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right\}$.

Solution. (a).

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n) &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n} - n) \cdot (\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n} + 1} = \frac{1}{2}. \end{aligned}$$

(b).

$$\lim_{n \rightarrow \infty} (2^n + 3^n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 3 \left[\left(\frac{2}{3} \right)^n + 1 \right]^{\frac{1}{n}} = 3 \cdot (0 + 1)^0 = 3.$$

Another solution: Since

$$3 = (3^n)^{\frac{1}{n}} \leq (2^n + 3^n)^{\frac{1}{n}} \leq (3^n + 3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3 \quad \text{and} \quad \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} \cdot 3 = 3,$$

$$\lim_{n \rightarrow \infty} (2^n + 3^n)^{\frac{1}{n}} = 3 \quad \text{by the Squeeze Theorem.}$$

(c).

$$\lim_{n \rightarrow \infty} \sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}} = \lim_{n \rightarrow \infty} \sqrt[4]{\frac{1 + 2\frac{n^5}{n!} + \frac{\ln n}{n!}}{1 + \frac{5^n}{n!} + 3\frac{n}{n!}}} = \sqrt[4]{\frac{1 + 0 + 0}{1 + 0 + 0}} = 1.$$

(d).

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{3n}{3n-1} \right)^{2n+\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{3n-1}{3n} \right)^{2n+\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{1}{\left[\left(1 + \frac{-1}{3n} \right)^{3n} \right]^{\frac{2n+\sqrt{n}}{3n}}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\frac{2 + \frac{1}{\sqrt{n}}}{\left[\left(1 + \frac{-1}{3n} \right)^{3n} \right]^3}} = \frac{1}{(e^{-1})^{\frac{2}{3}}} = e^{\frac{2}{3}}. \end{aligned}$$

(e). Note that

$$-\frac{n^{50} \cdot 50^n}{n!} \leq \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \leq \frac{n^{50} \cdot 50^n}{n!}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{n^{50} \cdot 50^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^{50}}{2^n} \cdot \frac{2^n \cdot 50^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^{50}}{2^n} \frac{100^n}{n!} = 0 \cdot 0 = 0,$$

$$\lim_{n \rightarrow \infty} \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} = 0 \quad \text{by the Squeeze Theorem.}$$

□

Question 3. [3 points, 1 for each part]

(a) If $\{a_n\}$ is convergent, show that $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$.

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geq 1$. Assume that $\{a_n\}$ is convergent, find its limit.

(c) Find the limit of the sequence

$$\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \dots \right\}.$$

Solution. (a). Let $A = \lim_{n \rightarrow \infty} a_n$. Given any $\epsilon > 0$, there exists N such that

$$|a_n - A| < \epsilon$$

for all $n > N$ (by $\epsilon - N$ definition). Write b_n for a_{n+1} , that is $b_n = a_{n+1}$. For all $n > N$,

$$|b_n - A| = |a_{n+1} - A| < \epsilon$$

because $n + 1 > n > N$. From $\epsilon - N$ definition, $\lim_{n \rightarrow \infty} b_n = A = \lim_{n \rightarrow \infty} a_n$, that is,

$$\lim_{n \rightarrow \infty} a_{n+1} = A = \lim_{n \rightarrow \infty} a_n.$$

(b). Let $A = \lim_{n \rightarrow \infty} a_n$. Then

$$A = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + a_n} = \frac{1}{1 + \lim_{n \rightarrow \infty} a_n} = \frac{1}{1 + A}.$$

Thus $A(1 + A) = 1$ or $A^2 + A - 1 = 0$. It follows that

$$A = \frac{-1 \pm \sqrt{5}}{2}.$$

Next we show that $a_n > 0$ by induction. When $n = 1$, $a_1 = 1 > 0$. Suppose that $a_n > 0$. Then $a_{n+1} = 1/(1 + a_n) > 0$. The induction is finished and so $a_n > 0$ for all n . It follows that $A = \lim_{n \rightarrow \infty} a_n \geq 0$. The value $\frac{-1 - \sqrt{5}}{2}$ is rejected because $A \geq 0$, and so

$$A = \frac{-1 + \sqrt{5}}{2}.$$

(c). From the sequence, we see that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. We show that $\lim_{n \rightarrow \infty} a_n$ exists:

First we prove by induction that $0 \leq a_n \leq 2$. When $n = 1$, $0 \leq a_1 \leq 2$ holds. Suppose that $0 \leq a_n \leq 2$. Then

$$0 \leq \sqrt{2a_n} = a_{n+1} \leq \sqrt{2 \cdot 2} = 2.$$

The induction is finished and so $0 \leq a_n \leq 2$ for all n .

Next since $0 \leq a_n \leq 2$, $a_{n+1} = \sqrt{2a_n} \geq \sqrt{a_n \cdot a_n} = a_n$ for all n . Thus $\{a_n\}$ is monotone increasing. By monotone convergence theorem, $\lim_{n \rightarrow \infty} a_n$ exists.

Finally let $A = \lim_{n \rightarrow \infty} a_n$. Then

$$A = \lim_{n \rightarrow \infty} a_{n+1} = \sqrt{2 \lim_{n \rightarrow \infty} a_n} = \sqrt{2A}$$

and so $A^2 = 2A$ or $A = 0, 2$. Since $\{a_n\}$ is monotone increasing, $a_n \geq a_1 = \sqrt{2}$ for all n . It follows that $A = \lim_{n \rightarrow \infty} a_n \geq \sqrt{2}$. Thus 0 is rejected and so $A = 2$. \square