

## Take-home Exam 2

**Question 1**[2 points, 1 for each part]

Let  $a_1$  and  $b_1$  be positive numbers with  $a_1 > b_1$ . Let  $a_2 = \frac{a_1 + b_1}{2}$  be their arithmetic mean and let  $b_2 = \sqrt{a_1 b_1}$  be their geometric mean. Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n}.$$

- (a) Show by mathematical induction that  $a_n > a_{n+1} > b_{n+1} > b_n$ .  
 (b) Prove that  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ .

*Solution.* (a). First we check the case  $n = 1$ . Since  $a_1 > b_1$ ,

$$a_1 > \frac{a_1 + b_1}{2} = a_2 > \sqrt{a_1 b_1} = b_2 > b_1,$$

where  $\frac{a_1 + b_1}{2} > \sqrt{a_1 b_1}$  because

$$a_1 + b_1 - 2\sqrt{a_1 b_1} = \left(\sqrt{a_1} - \sqrt{b_1}\right)^2 > 0.$$

Suppose that the statement holds for  $n$ , that is,

$$a_n > a_{n+1} > b_{n+1} > b_n.$$

Since  $a_{n+1} > b_{n+1}$ ,

$$a_{n+1} > \frac{a_{n+1} + b_{n+1}}{2} = a_{n+2} > \sqrt{a_{n+1} b_{n+1}} = b_{n+2} > b_{n+1},$$

where  $\frac{a_{n+1} + b_{n+1}}{2} > \sqrt{a_{n+1} b_{n+1}}$  because

$$a_{n+1} + b_{n+1} - 2\sqrt{a_{n+1} b_{n+1}} = \left(\sqrt{a_{n+1}} - \sqrt{b_{n+1}}\right)^2 > 0.$$

The induction is finished and so the statement holds for all  $n$ .

(b). From (a), the sequence  $\{a_n\}$  is monotone decreasing and bounded below and the sequence  $\{b_n\}$  is monotone increasing and bounded above. Thus  $\lim_{n \rightarrow \infty} a_n$  and  $\lim_{n \rightarrow \infty} b_n$  exist. Let  $A = \lim_{n \rightarrow \infty} a_n$  and let  $B = \lim_{n \rightarrow \infty} b_n$ . Then

$$A = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{a_n + b_n}{2} = \frac{\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n}{2} = \frac{A + B}{2}$$

and so  $2A = A + B$  or  $A = B$ , that is,  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ . □

**Question 2.** [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

- (a)  $\left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\}$ .  
 (b)  $\{[1.5 + (-1)^n]^n\}$ .

$$(c) \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} \right\}$$

*Solution.* (a).

$$\frac{n + (-1)^n n^2}{n^2 + 1} = \frac{\frac{1}{n} + (-1)^n}{1 + \frac{1}{n^2}} = \begin{cases} = \frac{\frac{1}{2k} + 1}{1 + \frac{1}{(2k)^2}} & n = 2k \\ = \frac{\frac{1}{2k+1} - 1}{1 + \frac{1}{(2k+1)^2}} & n = 2k + 1. \end{cases}$$

Since

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{2k} + 1}{1 + \frac{1}{(2k)^2}} = \frac{1}{1} = 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\frac{1}{2k+1} - 1}{1 + \frac{1}{(2k+1)^2}} = \frac{-1}{1} = -1,$$

the set of the subsequential limits is  $\{-1, 1\}$ , and so

$$\overline{\lim}_{n \rightarrow \infty} \frac{n + (-1)^n n^2}{n^2 + 1} = 1 \quad \text{and} \quad \underline{\lim}_{n \rightarrow \infty} \frac{n + (-1)^n n^2}{n^2 + 1} = -1.$$

(b).

$$[1.5 + (-1)^n]^n = \begin{cases} 2.5^{2k} & n = 2k \\ 0.5^{2k+1} & n = 2k + 1. \end{cases}$$

Since  $\lim_{k \rightarrow \infty} 2.5^{2k} = +\infty$  and  $\lim_{k \rightarrow \infty} 0.5^{2k+1} = 0$ ,

$$\overline{\lim}_{n \rightarrow \infty} [1.5 + (-1)^n]^n = +\infty \quad \text{and} \quad \underline{\lim}_{n \rightarrow \infty} [1.5 + (-1)^n]^n = 0.$$

(c). Let  $n = 16k + r$ , where  $k$  and  $r$  are integers with  $r = 0, 1, \dots, 15$  and  $k \geq 0$ . Then

$$1 + \sin \frac{n\pi}{8} = 1 + \sin\left(2k\pi + \frac{r\pi}{8}\right) = 1 + \sin \frac{r\pi}{8}.$$

If  $r \neq 12$ , then  $1 + \sin \frac{r\pi}{8} \geq 1 - \frac{\sqrt{2}}{2}$ , and so, for  $r \neq 12$ ,

$$\begin{aligned} & \left(1 + \frac{1}{16k+r}\right) \cdot \left(1 - \frac{\sqrt{2}}{2}\right)^{\frac{1}{16k+r}} \\ & \leq \left(1 + \frac{1}{16k+r}\right) \left(1 + \sin \frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}} \\ & \leq \left(1 + \frac{1}{16k+r}\right) \cdot 2^{\frac{1}{16k+r}}. \end{aligned}$$

Since

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k+r}\right) \cdot \left(1 - \frac{\sqrt{2}}{2}\right)^{\frac{1}{16k+r}} &= 1 \cdot \left(1 - \frac{\sqrt{2}}{2}\right)^0 = 1 \\ \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k+r}\right) \cdot 2^{\frac{1}{16k+r}} &= 1 \cdot 2^0 = 1, \\ \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k+r}\right) \left(1 + \sin \frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}} &= 1 \end{aligned}$$

for  $r \neq 12$  by the Squeeze Theorem. When  $r = 12$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k+r}\right) \left(1 + \sin \frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}} \\ = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k+r}\right) (1-1)^{\frac{1}{16k+r}} = \lim_{k \rightarrow \infty} 0 = 0. \end{aligned}$$

Thus the set of subsequential limits is  $\{0, 1\}$  and so

$$\underline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} = 0 \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} = 1.$$

*Another solution of Part (c).*

$$\text{Since } -1 \leq \sin \frac{n\pi}{8} \leq 1,$$

$$(1) \quad 0 \leq \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} \leq \left(1 + \frac{1}{n}\right) 2^{\frac{1}{n}}.$$

Thus

$$0 = \lim_{n \rightarrow \infty} 0 = \underline{\lim}_{n \rightarrow \infty} 0 \leq \underline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}}.$$

On the other hand, when  $n = 16k - 4$  for  $k = 1, 2, \dots$ ,

$$\sin \frac{n\pi}{8} = \sin \frac{(16k-4)\pi}{8} = \sin \left(2k\pi - \frac{\pi}{2}\right) = -1$$

and

$$\begin{aligned} &\underline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} \\ &\leq \lim_{k \rightarrow \infty} \left(1 + \frac{1}{16k-4}\right) \left(1 + \sin \frac{(16k-4)\pi}{8}\right)^{\frac{1}{16k-4}} = \lim_{k \rightarrow \infty} 0 = 0. \end{aligned}$$

It follows that

$$\underline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} = 0.$$

From Equation (1),

$$\overline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} \leq \overline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) 2^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) 2^{\frac{1}{n}} = 1.$$

On the other hand, by choosing  $n = 8k$ ,

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} \\ & \geq \lim_{k \rightarrow \infty} \left(1 + \frac{1}{8k}\right) \left(1 + \sin \frac{8k\pi}{8}\right)^{\frac{1}{8k}} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{8k}\right) = 1. \end{aligned}$$

Thus

$$\overline{\lim}_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \left(1 + \sin \frac{n\pi}{8}\right)^{\frac{1}{n}} = 1.$$

□

**Question 3** [5 points, 1 for each part]

Determine the convergence or divergence of each of the following series. Justify your answers.

- (a)  $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1}$ .
- (b)  $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$ .
- (c)  $\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$ .
- (d)  $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$ .
- (e)  $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k}$ .

*Solution.* (a). Since

$$\lim_{k \rightarrow \infty} \frac{\frac{\sqrt{k}}{k^2 + 2k - 1}}{\frac{1}{k^{\frac{3}{2}}}} = \lim_{k \rightarrow \infty} \frac{1}{1 + 2/k - 1/k^2} = 1,$$

and  $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$  converges by the  $p$ -series, the series converges by the limit comparison test.

(b). Let  $f(x) = \frac{1}{x(2 + \ln x)}$ . Then  $f(x)$  is positive and monotone decreasing on  $[1, +\infty)$ . Since

$$\begin{aligned} \int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x(2 + \ln x)} dx \\ &= \int_2^{\infty} \frac{dy}{y} = \ln y \Big|_2^{\infty} = +\infty \end{aligned}$$

diverges, the series  $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$  diverges by the integral test.

(c). Let  $a_n = 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$ . Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} 6 \left(1 - \frac{2}{n+1}\right)^n = \lim_{n \rightarrow \infty} 6 \frac{\left(1 - \frac{2}{n+1}\right)^{n+1}}{1 - \frac{2}{n+1}} = 6e^{-2} = \frac{6}{e^2} < 1$$

and so the series is convergent by the root test.

(d). Let  $a_n = \frac{n^n}{3^n \cdot n!}$ . Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1} \cdot 3^n \cdot n!}{3^{n+1} \cdot (n+1)! \cdot n^n} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n \cdot (n+1)}{3 \cdot (n+1)} = \frac{e}{3} < 1, \end{aligned}$$

the series is convergent by the ratio test.

(e). Let

$$a_k = \frac{\sqrt{k+1} - \sqrt{k}}{k} = \frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{k(\sqrt{k+1} + \sqrt{k})} = \frac{1}{k(\sqrt{k+1} + \sqrt{k})}$$

and let  $b_k = \frac{1}{k^{\frac{3}{2}}}$ . Then

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\frac{1}{k(\sqrt{k+1} + \sqrt{k})}}{\frac{1}{k^{\frac{3}{2}}}} = \lim_{k \rightarrow \infty} \frac{k^{\frac{3}{2}}}{k(\sqrt{k+1} + \sqrt{k})} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{1 + 1/k} + 1} = \frac{1}{2}.$$

Since  $\sum_{k=1}^{\infty} b_k$  converges by the  $p$ -series, the series converges by the limit comparison test.  $\square$