

Take-home Exam 3

Question 1. [2 points]

- (a) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, prove that $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.
 (b) If a and b are positive real numbers, prove that

$$\sum_{k=1}^{\infty} \frac{1}{(ak + b)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof. (a). Suppose that $\sum_{n=1}^{\infty} (a_n + b_n)$ converges. Since $\sum_{n=1}^{\infty} a_n$ converges by the assumption,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} [(a_n + b_n) - a_n]$$

converges, which contradicts to the assumption that $\sum_{n=1}^{\infty} b_n$ diverges. Hence $\sum_{n=1}^{\infty} (a_n + b_n)$ converges.

- (b). Let $a_k = \frac{1}{(ak + b)^p}$ and let $b_k = \frac{1}{k^p}$. Then

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{\frac{1}{(ak + b)^p}}{\frac{1}{k^p}} = \lim_{k \rightarrow \infty} \frac{k^p}{(ak + b)^p} = \lim_{k \rightarrow \infty} \frac{1}{(a + b/k)^p} = \frac{1}{a^p} \neq 0, \infty.$$

By the p -series, $\sum_{k=1}^{\infty} b_k$ converges if $p > 1$ and diverges if $p \leq 1$. Thus, by the limit comparison test, a_k converges if $p > 1$ and diverges if $p \leq 1$. \square

Question 2. [3 points, 1 for each part] Test the series for convergence or divergence.

- (a) $\sum_{k=1}^{\infty} (-1)^k 2^{1/k}$.
 (b) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(k+1)(k+2)}$.
 (c) $\sum_{k=1}^{\infty} (\sqrt[k]{2} - 1)$.

Solution. (a). Divergence. Since $\lim_{k \rightarrow \infty} 2^{1/k} = 1$, the limit $\lim_{k \rightarrow \infty} (-1)^k 2^{1/k}$ does not exist and so, by divergence test, the series diverges.

(b). Convergence. Let $b_k = \frac{k}{(k+1)(k+2)}$. Then $\{b_k\}$ is monotone decreasing because

$$\begin{aligned} b_k \geq b_{k+1} &\iff \frac{k}{(k+1)(k+2)} \geq \frac{(k+1)}{(k+2)(k+3)} \\ &\iff k(k+3) \geq (k+1)^2 \\ &\iff k^2 + 3k \geq k^2 + 2k + 1 \iff k \geq 1. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} b_k = \lim_{k \rightarrow \infty} \frac{1}{(1+1/k)(k+2)} = 0$, by the alternating series test, the series converges.

(c). Let $a_k = \sqrt[k]{k} - 1$ and let $b_k = \frac{1}{k}$. Then

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \lim_{k \rightarrow \infty} \frac{2^{1/k} - 1}{1/k} \stackrel{x=1/k}{=} \lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \lim_{x \rightarrow 0} \frac{2^x \ln 2}{1} = \ln 2.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=1}^{\infty} (\sqrt[k]{2} - 1)$ diverges. \square

Question 3 [5 points, 1 for each part]

Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

- (a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k} + 1}$.
- (b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2 + (-1)^k}$.
- (c) $\sum_{k=1}^{\infty} \frac{\sin kt}{k^2 + 3}$, $t \in \mathbb{R}$.
- (d) $\sum_{k=2}^{\infty} \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k} + 1}$.
- (e) $\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{(k+1)^k}$.

Solution. (a). Conditional convergence. Let $a_k = \frac{1}{\sqrt{k} + 1}$. Then $\{a_k\}$ is positive, monotone decreasing and $\lim_{k \rightarrow \infty} a_k = 0$. By the alternating series test, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k} + 1}$

converges. Since $\frac{1}{\sqrt{k} + 1} \geq \frac{1}{\sqrt{k} + \sqrt{k}} = \frac{1}{2\sqrt{k}}$ and $\sum_{k=1}^{\infty} \frac{1}{2\sqrt{k}} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by

the p -series, the series $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k} + 1} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + 1}$ diverges.

(b). Absolute convergence. Since

$$\left| \frac{(-1)^k}{2k^2 + (-1)^k} \right| = \frac{1}{2k^2 + (-1)^k} \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p -series, the series $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{2k^2 + (-1)^k} \right|$ converges by the comparison test.

(c). Absolute convergence. Since

$$\left| \frac{\sin kt}{k^2 + 3} \right| \leq \frac{1}{k^2 + 3} \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p -series, the series $\sum_{k=1}^{\infty} \left| \frac{\sin kt}{k^2 + 3} \right|$ converges by the comparison test.

(d). Conditional convergence. Let $a_k = \frac{\ln(\ln k)}{\sqrt{\ln k} + 1}$. Then $a_k \geq 0$ for all k . To see $\{a_k\}$ eventually monotone decreasing, let $f(x) = \frac{\ln(\ln x)}{\sqrt{\ln x} + 1}$. Then

$$\begin{aligned} f'(x) &= \frac{\frac{1}{\ln x} \cdot \frac{1}{x} (\sqrt{\ln x} + 1) - \ln(\ln x) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\ln x}} \cdot \frac{1}{x}}{(\sqrt{\ln x} + 1)^2} \\ &= \frac{2(\sqrt{\ln x} + 1) - \ln(\ln x) \cdot \sqrt{\ln x}}{2x \ln x (\sqrt{\ln x} + 1)^2} = \frac{2 \left(1 + \frac{1}{\sqrt{\ln x}} \right) - \ln(\ln x)}{2x (\ln x)^{\frac{3}{2}} (\sqrt{\ln x} + 1)^2} \end{aligned}$$

is **eventually** negative because the denominator is positive when $x > 1$, and the numerator is negative when x is large, say when $x > e^{e^4}$,

$$2 \left(1 + \frac{1}{\sqrt{\ln x}} \right) - \ln(\ln x) < 2(1 + 1) - \ln(\ln x) = 4 - \ln(\ln x) < 0.$$

Thus $\{a_k\}$ is **eventually** monotone decreasing. Now

$$\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{\ln(\ln k)}{\sqrt{\ln k} + 1} \stackrel{x = \ln k}{=} \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x} + 1} = \lim_{x \rightarrow \infty} \frac{\frac{\ln x}{x^{\frac{1}{2}}}}{1 + \frac{1}{\sqrt{x}}} = \frac{0}{1} = 0.$$

By the alternating series test, the series $\sum_{k=2}^{\infty} \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k} + 1}$ converges. Since

$$\left| \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k} + 1} \right| = \frac{\ln(\ln k)}{\sqrt{\ln k} + 1} \geq \frac{1}{\sqrt{\ln k} + 1} \geq \frac{1}{\sqrt{\ln k} + \sqrt{\ln k}} = \frac{1}{2\sqrt{\ln k}} \geq \frac{1}{2\sqrt{k}}$$

for $\ln(\ln k) \geq 1$ (i.e., $k \geq e^e$) and the p -series $\sum_{k=2}^{\infty} \frac{1}{2\sqrt{k}} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges, the series

$$\sum_{k=2}^{\infty} \left| \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k} + 1} \right| \text{ diverges.}$$

(e). Divergence. Since

$$\lim_{k \rightarrow \infty} \frac{k^k}{(k+1)^k} = \lim_{k \rightarrow \infty} \frac{1}{\left(\frac{k+1}{k}\right)^k} = \lim_{k \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{k}\right)^k} = \frac{1}{e},$$

the limit $\lim_{k \rightarrow \infty} \frac{(-1)^k k^k}{(k+1)^k}$ does not exist because it has two subsequential limits $\pm \frac{1}{e}$. By

the divergence test, the series $\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{(k+1)^k}$ diverges. \square