2003/2004 Semester I MA2108 Advanced Calculus II

Take-home Exam 4

Question 1 [4 points, 1 for each part]
For each of the following sequence of functions, determine whether is converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

(a) $\left\{\left(1-\dfrac{x^2}{n}\right)^{nx}\right\}, \quad x \in \mathbb{R}.$

(b) $\{(\cos x)^{2n}\}, \quad x \in \mathbb{R}.$

(c) $\left\{\dfrac{\sin nx}{\cos nx + nx}\right\}, \quad x \in [1, +\infty).$

(d) $\{f_n(x)\}, \quad f_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}, \quad x \in \mathbb{R}.$

Solution. (a).

$$\lim_{n\to\infty}\left(1-\frac{x^2}{n}\right)^{nx} = \lim_{n\to\infty}\left[\left(1-\frac{x^2}{n}\right)^{n}\right]^{x} = \left(e^{-x^2}\right)^{x} = e^{-x^3}.$$

(b). When $x = k\pi$, then $\cos x = (-1)^k$ and so $\lim\limits_{n\to\infty}(\cos x)^{2n} = \lim\limits_{n\to\infty} 1 = 1$ for $x = k\pi$. When $x \neq k\pi$, the $|\cos x| < 1$ and so $\lim\limits_{n\to\infty}(\cos x)^{2n} = 0$ in this case. Thus

$$\lim_{n\to\infty}(\cos x)^{2n} = \begin{cases} 0 & x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots \\ 1 & x = k\pi \quad \text{for some} \quad k \in \mathbb{Z}. \end{cases}$$

(c).

$$\lim_{n\to\infty}\frac{\sin nx}{\cos nx + nx} = \lim_{n\to\infty}\frac{\dfrac{\sin nx}{n}}{\dfrac{\cos nx}{n} + x} = 0,$$

where $\lim\limits_{n\to\infty}\dfrac{\sin nx}{n} = \lim\limits_{n\to\infty}\dfrac{\cos nx}{n} = 0$, by the Squeeze Theorem, because

$$0 \leq \left|\frac{\sin nx}{n}\right|, \left|\frac{\cos nx}{n}\right| \leq \frac{1}{n}$$

and $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$.

(d). When $x = 0$, $f_n(x) = 0$. In this case $\lim\limits_{n\to\infty} f_n(x) = 0$. When $x \neq 0$,

$$f_n(x) = \sum_{k=0}^{n}\frac{x^2}{(1+x^2)^k}$$

$$= x^2\left[1 + \frac{1}{1+x^2} + \cdots + \left(\frac{1}{1+x^2}\right)^n\right]$$

$$= x^2 \frac{1 - \left(\dfrac{1}{1+x^2}\right)^{n+1}}{1 - \dfrac{1}{1+x^2}}$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^2 \frac{1 - \left(\dfrac{1}{1+x^2}\right)^{n+1}}{1 - \dfrac{1}{1+x^2}} = x^2 \frac{1}{1 - \dfrac{1}{1+x^2}} = 1 + x^2$$

because the positive number $\dfrac{1}{1+x^2} < 1$ for $x \neq 0$. Thus

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x = 0 \\ 1 + x^2 & x \neq 0. \end{cases}$$

□

Question 2. [6 points, 1 for each part] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $F_n(x) = \dfrac{x^n}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(b) $F_n(x) = \dfrac{x^n}{1+x^n}, \quad x \in [0, 1].$

(c) $F_n(x) = x + \dfrac{x}{n} \sin nx, \quad x \in [-a, a], \quad a > 0.$

(d) $F_n(x) = x + \dfrac{x}{n} \sin nx, \quad x \in \mathbb{R}.$

(e) $F_n(x) = \dfrac{x^n \sin nx}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(f) $F_n(x) = nx\left(1 - x^2\right)^n, \quad x \in [0, 1].$

Solution. (a). Uniform convergence.

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \qquad 0 \le x \le \frac{1}{2}$$

$$\begin{aligned} T_n &= \sup_{0 \le x \le \frac{1}{2}} |F_n(x) - F(x)| \\ &= \sup_{0 \le x \le \frac{1}{2}} \frac{x^n}{1+x^n} \\ &\le \sup_{0 \le x \le \frac{1}{2}} \frac{x^n}{1} = \left(\frac{1}{2}\right)^n \end{aligned}$$

Since $\lim_{n\to\infty} \left(\dfrac{1}{2}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(b). NOT uniform convergence.

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & 0 \le x < 1 \\ \frac{1}{1+1} = \frac{1}{2} & x = 1. \end{cases}$$

Since each $F_n(x) = \dfrac{x^n}{1+x^n}$ is continuous on $[0,1]$ and its limiting function $F(x)$ is NOT continuous on $[0,1]$, the sequence of functions does not converge uniformly on $[0,1]$.

For (c) and (d), $F(x) = \lim\limits_{n\to\infty} x + \dfrac{x}{n}\sin nx = x$ because $\lim\limits_{n\to\infty} \dfrac{\sin nx}{n} = 0$.

(c). Uniform convergence.

$$\begin{aligned}
T_n &= \sup_{x\in[-a,a]} |F_n(x) - F(x)| \\
&= \sup_{x\in[-a,a]} \left| x + \frac{x}{n}\sin nx - x \right| \\
&= \sup_{x\in[-a,a]} \left| \frac{x}{n}\sin nx \right| \\
&\le \sup_{x\in[-a,a]} \frac{|x|}{n} = \frac{a}{n}.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \dfrac{a}{n} = 0$, by the Squeeze Theorem, $\lim\limits_{n\to\infty} T_n = 0$ and so the sequence of functions converges uniformly.

(d). NOT uniform convergence.

$$T_n = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = \sup_{x\in\mathbb{R}} \left| x + \frac{x}{n}\sin nx - x \right| = \sup_{x\in\mathbb{R}} \frac{|x|\cdot|\sin nx|}{n}$$

By choosing $x = 2n\pi + \dfrac{\pi}{2n}$, we have

$$\begin{aligned}
T_n &\ge \frac{\left|2n\pi + \frac{\pi}{2n}\right| \cdot \left|\sin n(2n\pi + \frac{\pi}{2n})\right|}{n} \\
&= \frac{(2n\pi + \frac{\pi}{2n}) \cdot \left|\sin(2n^2\pi + \frac{\pi}{2})\right|}{n} \\
&= \frac{(2n\pi + \frac{\pi}{2n}) \cdot 1}{n} \\
&= 2\pi + \frac{\pi}{2n^2} \ge 2\pi.
\end{aligned}$$

Thus T_n does not tend to zero as n tends to ∞, and so the sequence of functions does not converge uniformly.

(e). Uniform convergence. Since

$$\left| \frac{x^n \sin nx}{1+x^n} \right| \le \left| \frac{x^n}{1+x^n} \right|$$

and $\lim\limits_{n\to\infty} \dfrac{x^n}{1+x^n} = 0$ for $0 \le x \le \dfrac{1}{2}$, we have

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n \sin nx}{1+x^n} = 0$$

by the Squeeze Theorem. Note that

$$\begin{aligned} T_n = \sup_{0\le x\le \frac{1}{2}} |F_n(x) - F(x)| = \sup_{0\le x\le \frac{1}{2}} \frac{x^n|\sin nx|}{1+x^n} \\ \le \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1+x^n} \\ \le \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1} \\ = \left(\frac{1}{2}\right)^n. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, $\lim\limits_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(f). NOT uniform convergence. When $x = 0, 1$, $F_n(x) = 0$ and so $F(x) = \lim\limits_{n\to\infty} F_n(x) = 0$ for $x = 0, 1$. When $0 < x < 1$, then $0 < 1 - x^2 < 1$ or $\dfrac{1}{1-x^2} > 1$. Thus, for $0 < x < 1$,

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} nx\left(1-x^2\right)^n = \lim_{n\to\infty} \frac{n}{\left(\frac{1}{1-x^2}\right)^n} \cdot x = 0$$

by the Standard limits. Thus $F(x) = 0$ for all $0 \le x \le 1$. Now

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} F_n(x) = \sup_{0\le x\le 1} nx\left(1-x^2\right)^n.$$

We find the maximum of $F_n(x)$ on $[0, 1]$.

$$\begin{aligned} F_n'(x) = n\left(1-x^2\right)^n + nx\cdot n\left(1-x^2\right)^{n-1}\cdot(-2x) = 0 \\ \Longrightarrow n\left(1-x^2\right)^{n-1}\left(1-x^2-2nx^2\right) = n\left(1-x^2\right)^{n-1}\left(1-(2n+1)x^2\right) = 0 \\ \Longrightarrow x = 0, 1, \frac{1}{\sqrt{2n+1}}. \end{aligned}$$

Now

$$F_n(0) = F_n(1) = 0, \qquad F_n\left(\frac{1}{\sqrt{2n+1}}\right) = n\cdot\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n$$

Thus $T_n = F_n\left(\dfrac{1}{\sqrt{2n+1}}\right) = n\cdot\dfrac{1}{\sqrt{2n+1}}\left(1-\dfrac{1}{2n+1}\right)^n$ and

$$\begin{aligned} \lim_{n\to\infty} T_n = \lim_{n\to\infty} n\cdot\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n \\ = \lim_{n\to\infty} \sqrt{n}\cdot\frac{1}{\sqrt{2+1/n}}\cdot\left[\left(1+\frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{n}{2n+1}} \\ = \infty\cdot\frac{1}{2}\cdot\left(e^{-1}\right)^{\frac{1}{2}} = \infty. \end{aligned}$$

Hence the sequence of functions does not converge uniformly. □