Take-home Exam 5

Question 1. [6 points, 1 for each part] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \quad x \in [0, \infty).$

(b) $\displaystyle\sum_{k=1}^{\infty} e^{-kx} x^k, \quad x \in [0, \infty).$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}, \quad x \in [0, \infty).$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \quad x \in [-1, 1].$

(e) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right), \quad x \in [0, 1].$

(f) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{x}{2} \right)^k, \quad x \in (-2, 2).$

Solution. (a). Uniform convergence. Since

$$\left| \frac{k \sin kx}{k^3 + x^2} \right| \leq \frac{k}{k^3 + x^2} \leq \frac{k}{k^3} = \frac{1}{k^2}$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the series of functions converges uniformly by the Weierstrass M-test.

(b). Uniform convergence. Let $f_k(x) = e^{-kx} x^k$.

$$f'(x) = (-k) \cdot e^{-kx} \cdot x^k + e^{-kx} \cdot k \cdot x^{k-1} = k e^{-kx} x^{k-1} (1 - x) = 0$$
$$\Longrightarrow x = 0, 1.$$

Since $f_k(0) = 0$ and $\displaystyle\lim_{x \to \infty} f_k(x) = \lim_{x \to \infty} \frac{x^k}{e^{kx}} = 0$, the maximum of $|f_k(x)| = f_k(x)$ on $[0, \infty)$ is $f_k(1) = e^{-k}$, that is,

$$\left| e^{-kx} x^k \right| \leq e^{-k} \qquad x \in [0, \infty).$$

Since $\displaystyle\sum_{k=1}^{\infty} e^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{e} \right)^k$ converges by the geometric series, the series of functions converges uniformly by the Weierstrass M-test.

(c). Uniform convergence. Let $a_k = \dfrac{1}{k + x}$. Then, for $x \geq 0$, a_k is positive, monotone decreasing and $\displaystyle\lim_{k \to \infty} a_k = 0$. Let $S(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}$ and let $S_n(x) =$

$\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+x}$. By the alternating series estimation,

$$|S_n(x) - S(x)| \le a_{n+1} = \frac{1}{n+1+x}.$$

Thus

$$T_n = \sup_{x\ge 0} |S_n(x) - S(x)| \le \sup_{x\ge 0} \frac{1}{n+1+x} \le \frac{1}{n+1}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0$, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} T_n = 0$ and so the series of functions converges uniformly.

(d). Uniform convergence. Note that

$$\left|\frac{x^k}{1+k(\ln k)^2}\right| \le \frac{1}{1+k(\ln k)^2} \qquad x \in [0,1].$$

Since $\displaystyle\frac{1}{1+k(\ln k)^2} \le \frac{1}{k(\ln k)^2}$ for $k \ge 2$ and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges by the integral test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k(\ln k)^2}$ converges by the comparison test. From the inequality above, the series of functions converges uniformly by the Weierstrass M-test.

(e). NOT uniform convergence. Consider the partial sums

$$S_n(x) = \sum_{k=0}^{n} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{x+2}\right) + \left(\frac{1}{x+2} - \frac{1}{2x+2}\right) + \cdots + \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2}\right)$$

$$= \frac{1}{2} - \frac{1}{(n+1)x+2}.$$

Thus

$$S(x) = \sum_{n=0}^{\infty} \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2}\right) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{(n+1)x+2}\right)$$

$$= \begin{cases} 0 & x = 0 \\ \frac{1}{2} & x \ne 0. \end{cases}$$

Since each $S_n(x)$ is continuous on $[0,1]$ and $S(x)$ is NOT continuous on $[0,1]$, the series of functions does not converge uniformly.

(f). NOT uniform convergence. Note that

$$T_n = \sup_{-2<x<2} \left| \sum_{k=n+1}^{\infty} \left(\frac{x}{2}\right)^k \right|$$

$$= \sup_{-2<x<2} \left| \left(\frac{x}{2}\right)^{n+1} \left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots\right] \right|$$

$$= \sup_{-2<x<2} \frac{\left(\frac{|x|}{2}\right)^{n+1}}{1-\frac{x}{2}}$$

$$\xlongequal{\text{let } x\to 2^-} +\infty.$$

Since T_n does not tend to 0 as n tends to ∞, the series of functions does not converge uniformly. ☐

Question 2 [4 points, 1 for each part] Find the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k$.

(b) $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2} (x-1)^k$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}x^k$

(d) $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n^2}$.

(*Hint.* For part (d), you can use the root test to find the interval for which the series converges absolutely.)

Solution. (a). Note that $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k = \sum_{k=1}^{\infty} \frac{3^k \cdot 2^k}{k^3}\left(x+\frac{1}{2}\right)^k$. Let $a_k = \dfrac{3^k\cdot 2^k}{k^3}$. Then the radius of convergence

$$R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty}\left(\frac{3^k\cdot 2^k}{k^3}\right)^{\frac{1}{k}}} = \frac{1}{\lim\limits_{k\to\infty}\frac{6}{\left(\sqrt[k]{k}\right)^3}} = \frac{1}{6}.$$

(b). The radius of convergence

$$R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty}\left[\left(1-\frac{1}{k}\right)^{k^2}\right]^{\frac{1}{k}}} = \frac{1}{\lim\limits_{k\to\infty}\left(1-\frac{1}{k}\right)^k} = \frac{1}{e^{-1}} = e.$$

(c). The radius of convergence

$$R = \frac{1}{\lim\limits_{k\to\infty}\frac{|a_{k+1}|}{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty}\frac{((k+1)!)^2}{(2k+2)!}\cdot\frac{(2k)!}{(k!)^2}}$$

$$= \frac{1}{\lim\limits_{k\to\infty}\frac{(k+1)^2}{(2k+2)(2k+1)}} = \frac{1}{\lim\limits_{k\to\infty}\frac{(1+1/k)^2}{(2+2/k)(2+1/k)}} = 4.$$

(d). The radius of convergence is 2. Let $a_n = n\left(\frac{x}{2}\right)^{n^2}$. Then

$$l = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \left(\frac{|x|}{2}\right)^n = \begin{cases} 0 & |x| < 2 \\ 1 & |x| = 2 \\ +\infty & |x| > 2 \end{cases}$$

When $|x| < 2$, then the series $\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n^2}$ converges absolutely. When $|x| \geq 2$, the sequence $\{a_n\}$ does not tend to 0 because $|a_n| = n\left(\frac{|x|}{2}\right)^{n^2} \geq n$ if $|x| \geq 2$. By the Divergence Test, the series $\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n^2}$ diverges for $|x| \geq 2$. In conclusion, the series $\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n^2}$ converges if and only if $|x| < 2$ and so the radius of convergence is 2. □