

Take-home Exam 6

Question 1. [4 points, 1 for each part] For each of the following, determine all values of x for which the given series converges.

$$(a) \quad \sum_{k=1}^{\infty} \frac{(1-2x)^k}{k}.$$

$$(b) \quad \sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k.$$

$$(c) \quad \sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k}, \quad x \neq 1.$$

$$(d) \quad \sum_{k=1}^{\infty} \frac{1}{kx^k}, \quad x \neq 0.$$

Solution. (a). $\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k} \left(x - \frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{|-2|^k}{k}\right)^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{2}{\sqrt[k]{k}}} = \frac{1}{2}.$$

The ending points $x = x_0 \pm R = \frac{1}{2} \pm \frac{1}{2} = 0, 1$. When $x = 0$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

diverges by the p -series. When $x = 1$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the alternating series test. In conclusion, the interval of convergence is $(0, 1]$.

(b). $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{6^k}{k^3} \left(x + \frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{6^k}{k^3}\right)^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{6}{(\sqrt[k]{k})^3}} = \frac{1}{6}.$$

The ending points $x = x_0 \pm R = -\frac{1}{2} \pm \frac{1}{6} = -\frac{2}{3}, -\frac{1}{3}$. When $x = -\frac{2}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$$

converges by the alternating series test. When $x = -\frac{1}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

converges by the p -series. In conclusion, the interval of convergence is $[-\frac{2}{3}, -\frac{1}{3}]$.

(c). Let $t = \frac{3x}{2(1-x)}$. Then $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k} = \sum_{k=1}^{\infty} t^k$. By the geometric series, the series $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k}$ converges if and only if

$$\begin{aligned} |t| < 1 &\iff \left| \frac{3x}{2(1-x)} \right| < 1 \\ &\iff -1 < \frac{3x}{2(1-x)} < 1 \\ &\iff \begin{cases} \frac{3x}{2(1-x)} < 1 &\iff \frac{3x-2(1-x)}{2(1-x)} < 0 &\iff \frac{5x-2}{2(x-1)} > 0 &\iff x < \frac{2}{5} \text{ or } x > 1 \\ \frac{3x}{2(1-x)} > -1 &\iff \frac{3x+2(1-x)}{2(1-x)} > 0 &\iff \frac{x+2}{2(x-1)} < 0 &\iff -2 < x < 1 \end{cases} \\ &\iff -2 < x < \frac{2}{5}. \end{aligned}$$

Thus the answer is $-2 < x < \frac{2}{5}$.

(d). Let $t = \frac{1}{x}$. Then $\sum_{k=1}^{\infty} \frac{1}{kx^k} = \sum_{k=1}^{\infty} \frac{t^k}{k}$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k}}} = 1.$$

The ending points $t = x_0 \pm R = 0 \pm 1 = \pm 1$. When $t = 1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p -series. When $t = -1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. Thus the interval of convergence is

$$\begin{aligned} -1 \leq t < 1 &\iff -1 \leq \frac{1}{x} < 1 \\ &\iff \begin{cases} \frac{1}{x} < 1 &\iff \frac{1-x}{x} < 0 &\iff \frac{x-1}{x} > 0 &\iff x < 0 \text{ or } x > 1 \\ \frac{1}{x} \geq -1 &\iff \frac{1+x}{x} \geq 0 &\iff x > 0 \text{ or } x \leq -1 \\ &\iff x \leq -1 \text{ or } x > 1. \end{cases} \end{aligned}$$

Thus the answer is $x \leq -1$ or $x > 1$. □

Question 2. [4 points, 1 for each part] Using any applicable method, find the Taylor series of each of the following functions at the indicated point, and specify the interval on which the series converges to the function.

- (a) $f(x) = \cos x^2, \quad x_0 = 0.$
 (b) $f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad x_0 = 0.$
 (c) $f(x) = \sqrt{x}, \quad x_0 = 1.$
 (d) $f(x) = \frac{x^2}{1-x^2}, \quad x_0 = 0.$

Solution. (a). Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for $x \in \mathbb{R}$,

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

and the interval is $(-\infty, \infty)$.

(b). Since

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

for $-1 < x \leq 1$, we have

$$\begin{aligned} f(x) &= \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-x)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n - (-1)^{2n+1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} x^n \\ &= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots = 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \end{aligned}$$

for $-1 < x < 1$.

(c). By using the binomial series,

$$f(x) = \sqrt{x} = \sqrt{1+(x-1)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x-1)^n$$

for $|x-1| < 1$, that is, $0 < x < 2$.

(d). By using the geometric series,

$$\begin{aligned} f(x) &= \frac{x^2}{1-x^2} = x^2 \sum_{n=0}^{\infty} (x^2)^n = x^2 \sum_{n=0}^{\infty} x^{2n} \\ &= x^2 (1 + x^2 + x^4 + x^6 + \dots) \\ &= x^2 + x^4 + x^6 + x^8 + \dots = \sum_{n=1}^{\infty} x^{2n} \end{aligned}$$

for $|x^2| < 1$, that is, $|x| < 1$. □

Question 3. [2 points, 1 for each part]

(a) Use series to estimate the integral's value

$$\int_0^{0.1} \arctan x^2 dx$$

with an error of magnitude less than 10^{-8} .

(b) Let $f(x) = x^3 \sin x^9$. Find $f^{(30)}(0)$.

Solution. (a). From $\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$, we have

$$\begin{aligned} \arctan x^2 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-2}}{2n-1} \\ \implies \int_0^{\frac{1}{10}} \arctan x^2 dx &= \sum_{n=1}^{\infty} \int_0^{\frac{1}{10}} \frac{(-1)^{n+1} x^{4n-2}}{2n-1} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-1}}{(4n-1)(2n-1)} \Big|_0^{\frac{1}{10}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n-1)(2n-1)10^{4n-1}}. \end{aligned}$$

Let $a_n = \frac{1}{(4n-1)(2n-1)10^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. By the alternating series estimation, from

$$a_{n+1} = \frac{1}{(4n+3)(2n+1)10^{4n+3}} < 10^{-8},$$

we have $n \geq 1$. Thus

$$\int_0^{0.1} \arctan x^2 dx \approx \frac{1}{3 \cdot 1 \cdot 10^3} = \frac{1}{3000}$$

with error less than 10^{-8} .

(b). Since $\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$,

$$\begin{aligned} f(x) &= x^3 \sin x^9 = x^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{9(2n-1)}}{(2n-1)!} \\ &= x^3 \left(x^9 - \frac{x^{27}}{3!} + \frac{x^{45}}{5!} - \dots \right) \\ &= x^{12} - \frac{x^{30}}{3!} + \frac{x^{48}}{5!} - \dots \end{aligned}$$

By the definition of Taylor series,

$$\frac{f^{(30)}(0)}{30!} = -\frac{1}{3!}$$

and so $f^{(30)}(0) = -\frac{30!}{3!}$. □