2003/2004 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 1

Question 1(i). Observe that

$$\left|\frac{2n^2}{3n^2+2}-\frac{2}{3}\right| = \left|\frac{6n^2-2(3n^2+2)}{3(3n^2+2)}\right| = \frac{4}{9n^2+6} \le \frac{4}{9n^2}.$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{4}{9N^2} \le \epsilon$ or $N \ge \frac{2}{3\sqrt{\epsilon}}$. Then

$$\left|\frac{2n^2}{3n^2+2}-\frac{2}{3}\right| \le \frac{4}{9n^2} < \frac{4}{9N^2} \le \epsilon$$

for $n > N$ and hence the result. □

Question 1 (ii). Observe that

$$\left|\sqrt{\frac{n}{n+1}}-1\right| = \left|\frac{\sqrt{n}}{\sqrt{n+1}}-1\right| = \frac{\left|\sqrt{n}-\sqrt{n+1}\right|}{\sqrt{n+1}}$$

$$= \frac{\left|(\sqrt{n}-\sqrt{n+1})(\sqrt{n}+\sqrt{n+1})\right|}{\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})} = \frac{|n-(n+1)|}{\sqrt{n^2+n}+n+1} = \frac{1}{\sqrt{n^2+n}+n+1} \le \frac{1}{n}.$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{1}{N} \le \epsilon$ or $N \ge \frac{1}{\epsilon}$. Then

$$\left|\sqrt{\frac{n}{n+1}}-1\right| \le \frac{1}{n} < \frac{1}{N} \le \epsilon$$

for $n > N$ and hence the result. □

Question 2 (i). This statement is true. We prove the statement by contradiction. Suppose that $\{a_n - b_n\}$ is convergent. Let $c_n = a_n + b_n$. By Theorem 1.4.5, the sequence

$$\{c_n - a_n\} = \{(a_n + b_n) - a_n\} = \{b_n\}$$

is convergent. This contradicts to the assumption that $\{b_n\}$ is divergent and hence $\{a_n + b_n\}$ is divergent. □

Question 2 (ii). This statement is false. An counter-example is as follows. Let $a_n = \frac{1}{n}$ and let $b_n = n$. Then $\{a_n\}$ is convergent and $\{b_n\}$ is divergent. But the sequence $\{a_n b_n\} = \{1, 1, 1, \cdots\}$ is convergent. □

Question 3 (i).

$$\lim_{n\to\infty}\frac{n^2+5n^3-1}{3n^3+6n+4} = \lim_{n\to\infty}\frac{(n^2+5n^3-1)/n^3}{(3n^3+6n+4)/n^3} = \lim_{n\to\infty}\frac{\frac{1}{n}+5-\frac{1}{n^3}}{3+\frac{6}{n^2}+\frac{4}{n^3}} = \frac{0+5-0}{3+0+0} = \frac{5}{3}.$$

□

Question 3 (ii).

$$\lim_{n\to\infty}\frac{3^n+n^8}{2n^2+7^n}=\lim_{n\to\infty}\frac{(3^n+n^8)/7^n}{(2n^2+7^n)/7^n}=\lim_{n\to\infty}\frac{(\frac{3}{7})^n+\frac{n^8}{7^n}}{\frac{2n^2}{7^n}+1}=\frac{0+0}{0+1}=0.$$

□

Question 3 (iii).

$$\lim_{n\to\infty}\sqrt{\frac{n^4+4n^3+1}{n^3+2n^2}}=\lim_{n\to\infty}\sqrt{\frac{n^4}{n^3}\cdot\frac{(n^4+4n^3+1)/n^4}{(n^3+2n^2)/n^3}}=\lim_{n\to\infty}\sqrt{n}\cdot\sqrt{\frac{1+\frac{4}{n}+\frac{1}{n^4}}{1+\frac{2}{n}}}=+\infty.$$

□

Question 4 (i). Observe that

$$\frac{1}{2n}\le\frac{1+|\sin n|}{2n}\le\frac{2}{2n}=\frac{1}{n}.$$

Since $\lim_{n\to\infty}\frac{1}{2n}=\lim_{n\to\infty}\frac{1}{n}=0$, we have $\lim_{n\to\infty}\frac{1+|\sin n|}{2n}=0$ by the Squeeze Theorem.

□

Question 4 (ii). Since

$$0\le\frac{2n-5}{3n+1}\le\frac{2n}{3n}=\frac{2}{3}$$

for $n\ge 3$, we have

$$0\le\left(\frac{2n-5}{3n+1}\right)^n\le\left(\frac{2}{3}\right)^n$$

for $n\ge 3$. Since $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=\lim_{n\to\infty}0=0$, we have $\lim_{n\to\infty}\left(\frac{2n-5}{3n+1}\right)^n=0$ by the Squeeze Theorem. □

Question 5 (i). Since $\frac{e^n}{n^{100}}>0$ and $\lim_{n\to\infty}\frac{n^{100}}{e^n}=0$, we have $\lim_{n\to\infty}\frac{e^n}{n^{100}}=+\infty$ by the Reciprocal Rule □

Question 5 (ii). Since $\frac{n}{\ln\frac{1}{n+2}}=-\frac{n}{\ln(n+2)}<0$ and

$$\lim_{n\to\infty}-\frac{\ln(n+2)}{n}=\lim_{n\to\infty}-\frac{\frac{1}{n+2}}{1}=0,$$

we have $\lim_{n\to\infty}\frac{n}{\ln\frac{1}{n+2}}=-\infty$ by the Reciprocal Rule. □

Question 6 (a).

$$\lim_{n\to\infty}\sqrt[3]{\frac{2n^4+n+1}{16n^4+n^2+2}}=\lim_{n\to\infty}\sqrt[3]{\frac{(2n^4+n+1)/n^4}{(16n^4+n^2+2)/n^4}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{2+\frac{1}{n^3}+\frac{1}{n^4}}{16+\frac{1}{n^2}+\frac{2}{n^4}}} = \sqrt[3]{\frac{2+0+0}{16+0+0}} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}.$$

□

Question 6 (b).

$$\lim_{n\to\infty}\left(3+\ln\left(\cos\frac{1}{\sqrt{n}}\right)+\frac{n^2}{1.1^n}\right) = 3+\ln(\cos 0)+0 = 3.$$

□

Question 6 (c).

$$\lim_{n\to\infty}\frac{n^4+8^n}{9^n+n+8^n} = \lim_{n\to\infty}\frac{(n^4+8^n)/9^n}{(9^n+n+8^n)/9^n} = \lim_{n\to\infty}\frac{\frac{n^4}{9^n}+\left(\frac{8}{9}\right)^n}{1+\frac{n}{9^n}+\left(\frac{8}{9}\right)^n} = \frac{0+0}{1+0+0} = 0.$$

□

Question 6 (d).

$$\lim_{n\to\infty}\sqrt[n]{\frac{n}{3}} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{3^{\frac{1}{n}}} = \frac{1}{3^0} = 1.$$

□

Question 6 (e).

$$\lim_{n\to\infty}\left(\sqrt{3n-2}-\sqrt{3n-3}\right) = \lim_{n\to\infty}\frac{\left(\sqrt{3n-2}-\sqrt{3n-3}\right)\left(\sqrt{3n-2}+\sqrt{3n-3}\right)}{\sqrt{3n-2}+\sqrt{3n-3}}$$

$$= \lim_{n\to\infty}\frac{(3n-2)-(3n-3)}{\sqrt{3n-2}+\sqrt{3n-3}} = \lim_{n\to\infty}\frac{1}{\sqrt{3n-2}+\sqrt{3n-3}} = 0.$$

□

Question 6 (f). Since $0 \le \frac{3+(-1)^n}{5} \le \frac{4}{5}$, we have $0 \le \left(\frac{3+(-1)^n}{5}\right)^n \le \left(\frac{4}{5}\right)^n$. Since $\lim_{n\to\infty}\left(\frac{4}{5}\right)^n = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty}\left(\frac{3+(-1)^n}{5}\right)^n = 0$ by the Squeeze Theorem. □

Question 6 (g).

$$\lim_{n\to\infty}\frac{7^n+\ln n-n!}{n!+n^2} = \lim_{n\to\infty}\frac{(7^n+\ln n-n!)/n!}{(n!+n^2)/n!} = \lim_{n\to\infty}\frac{\frac{7^n}{n!}+\frac{\ln n}{n!}-1}{1+\frac{n^2}{n!}} = \frac{0+0-1}{1+0} = -1.$$

□

Question 6 (h).

$$\lim_{n\to\infty}\frac{n^{100}100^n}{n!} = \lim_{n\to\infty}\frac{n^{100}}{2^n}\cdot\frac{2^n\cdot 100^n}{n!} = \lim_{n\to\infty}\frac{n^{100}}{2^n}\cdot\frac{200^n}{n!} = 0\cdot 0 = 0.$$

□

Question 6 (i). Observe that

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$$

Since

$$-\frac{\ln n}{\sqrt{n}} \le \frac{(-1)^{n+1}\ln n}{\sqrt{n}} \le \frac{\ln n}{\sqrt{n}},$$

we have $\lim\limits_{n\to\infty} \dfrac{(-1)^{n+1}\ln n}{\sqrt{n}} = 0$ by the Squeeze Theorem. □

Question 6 (j).

$$\lim_{n\to\infty} (3^n + 4^n)^{\frac{1}{n}} = \lim_{n\to\infty} (4^n)^{\frac{1}{n}} \cdot \left(\left(\frac{3}{4}\right)^n + 1 \right)^{\frac{1}{n}} = \lim_{n\to\infty} 4 \left(\left(\frac{3}{4}\right)^n + 1 \right)^{\frac{1}{n}} = 4\cdot(0+1)^0 = 4.$$

□

Question 6 (k).

$$\lim_{n\to\infty} n \sin\frac{3}{n} = \lim_{n\to\infty} 3 \cdot \frac{\sin\frac{3}{n}}{\frac{3}{n}} = \lim_{x\to 0} 3 \cdot \frac{\sin x}{x} = 3 \cdot 1 = 3,$$

where $x = \dfrac{3}{n}$. □

Question 6 (l).

$$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^{2n} = \lim_{n\to\infty} \left(\left(1 + \frac{-\frac{1}{3}}{n}\right)^n \right)^2 = \left(e^{-\frac{1}{3}}\right)^2 = e^{-\frac{2}{3}}.$$

□

Question 6 (m).

$$\lim_{n\to\infty} \left(1 + \frac{1}{n^3}\right)^{n^3+2} = \lim_{n\to\infty} \left(1 + \frac{1}{n^3}\right)^{n^3} \cdot \left(1 + \frac{1}{n^3}\right)^2$$

$$= \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n^3}\right)^2 = e^1 \cdot (1+0)^2 = e,$$

where $m = n^3$. □

Question 7. Since S and T are bounded sets, there exist numbers m_1, m_2, M_1 and M_2 such that $m_1 \le x \le M_1$ for all $x \in S$ and $m_2 \le y \le M_2$ for all $y \in T$. Let z be any number in $S \cup T$. Then $z \in S$ or T. If $z \in S$, then $m_1 \le z \le M_1$. Otherwise, $z \in T$ and $m_2 \le z \le M_2$. In both cases, we have $\min\{m_1, m_2\} \le z \le \max\{M_1, M_2\}$ and so $S \cup T$ is bounded, where $\max\{M_1, M_2\}$ is an upper bound and $\min\{m_1, m_2\}$ is a lower bound. □

Question 8 (i). $\Longrightarrow$ Suppose that $\{a_n\}$ is bounded. There exist numbers m and M such that $m \le a_n \le M$ for all n. Thus $0 \le |a_n| \le \max\{|m|, |M|\}$ for all n and so $\{|a_n|\}$ is a bounded sequence, where $\max\{|m|, |M|\}$ is an upper bound and 0 is a lower bound.

$\Longleftarrow$ Suppose that $\{|a_n|\}$ is bounded. Then $\{|a_n|\}$ has an upper bound, that is there exists a number M such that $|a_n| \le M$ for all n. It follows that $-M \le a_n \le M$ for all n. Thus $\{a_n\}$ is a bounded sequence, where M is an upper bound and $-M$ is a lower bound. $\square$

Question 8 (ii). Since $\{b_n\}$ is bounded, the sequence $\{|b_n|\}$ is bounded by (i) and so there exists a positive number M such that $|b_n| \le M$ for all n. Given any $\epsilon > 0$, since $\lim_{n\to\infty} a_n = 0$, there exists a positive integer N such that $|a_n| = |a_n - 0| < \frac{\epsilon}{M}$ for all $n > N$. It follows that

$$|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| \le |a_n| \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $n > N$ and so $\lim_{n\to\infty} a_n b_n = 0$. $\square$