Solutions to Tutorial 4

Question 1 (a). Let $a_n = \dfrac{n^2-1}{2n^2+n}$. Then $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2} \neq 0$ and so the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{2n^2+n}$ is divergent by the divergence test. □

Question 1 (b). Let $a_n = \sin\dfrac{n\pi}{2}$. Then $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ and so $\lim\limits_{n\to\infty} a_n$ does not exist. Thus the series $\sum\limits_{n=1}^{\infty} \sin\dfrac{n\pi}{2}$ is divergent by the divergence test. □

Question 1 (c). Let $a_n = \dfrac{n^2+1+\ln n}{n+n^3+4}$ and let $b_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{n+n^3+4}{n^2+1+\ln n} = \lim_{n\to\infty}\frac{\frac{1}{n^2}+1+\frac{4}{n^3}}{1+\frac{1}{n^2}+\frac{\ln n}{n^2}} = \frac{0+1+0}{1+0+0} = 1.$$

Since $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ is divergent, so is $\sum\limits_{n=1}^{\infty}\dfrac{n^2+1+\ln n}{n+n^3+4}$ by the limit comparison test. □

Question 1 (d). Observe that

$$\frac{3+\sin n}{n^2} \leq \frac{4}{n^2}.$$

Since $\sum\limits_{n=1}^{\infty}\dfrac{4}{n^2} = 4\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is convergent by the p-series, the positive series $\sum\limits_{n=1}^{\infty}\dfrac{3+\sin n}{n^2}$ is convergent by the comparison test. □

Question 1 (e). Observe that

$$\frac{2^n+3}{3^{n+1}-n} \leq \frac{2^n+2^n}{3^{n+1}} = \frac{2^{n+1}}{3^n} = 2\left(\frac{2}{3}\right)^n$$

for $n \geq 2$. Since $\sum\limits_{n=1}^{\infty} 2\left(\dfrac{2}{3}\right)^n$ is convergent by the geometric series, the positive series $\sum\limits_{n=1}^{\infty}\dfrac{2^n+3}{3^{n+1}-n}$ is convergent by the comparison test. □

Question 1 (f). Let $a_n = \dfrac{2}{n^{1+\frac{1}{n}}}$ and let $b_n = \dfrac{1}{n}$. Observe that

$$\lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{n^{1+\frac{1}{n}}}{2} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$

Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ is divergent by the harmonic series, the positive series $\sum\limits_{n=1}^{\infty}\dfrac{2}{n^{1+\frac{1}{n}}}$ is divergent by the limit comparison test. □

Question 1 (g). Observe that

$$\frac{4+(-1)^n}{2n} \geq \frac{3}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{4+(-1)^n}{2n}$ is divergent by the comparison test. □

Question 1 (h). Observe that

$$\frac{1}{n(1+\ln n)^p} = \frac{(1+\ln n)^{-p}}{n} \geq \frac{1}{n}$$

for $p \leq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^p}$ is divergent for $p \leq 0$ by the comparison test. □

Question 1 (i). Observe that

$$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by the comparison test. □

Question 2(a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$ and so there exists a positive integer N such that $a_n = |a_n| = |a_n - 0| < 1$ for $n > N$. Since $a_n \geq 0$,

$$a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n$$

for $n > N$. By the comparison test, the positive series $\sum_{n=1}^{\infty} a_n^2$ is convergent. □

Another Solution of Question 2 (a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$. Let $b_n = a_n^2$. Then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} a_n = 0,$$

that is, $b_n << a_n$. Since $a_n \geq 0$, by limit comparison test, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2$ is convergent. □

Question 2 (b). Let $a_n = \dfrac{1}{n^2}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent by the p-series. $\square$

Question 3 (a). Let $f(x) = \dfrac{1}{x(1+\ln x)}$. Then $f(x)$ is a positive monotone decreasing function over $[1, +\infty)$. Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x(1+\ln x)}dx \xlongequal[y=\ln x]{} \int_0^{\infty} \frac{1}{1+y}dy = \ln(1+y)\Big|_0^{\infty} = +\infty.$$

is divergent, the series $\sum_{n=1}^{\infty} \dfrac{1}{n(1+\ln n)}$ is divergent by the integral test. $\square$

Question 3 (b). Let $f(x) = \dfrac{1}{x[1+(\ln x)^2]}$. Then $f(x)$ is a positive monotone decreasing function over $[1, +\infty)$. Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x[1+(\ln x)^2]}dx \xlongequal[y=\ln x]{} \int_0^{\infty} \frac{1}{1+y^2}dy = \arctan y\Big|_0^{\infty} = \frac{\pi}{2}$$

is convergent, the series $\sum_{n=1}^{\infty} \dfrac{1}{n(1+\ln n)}$ is convergent by the integral test. $\square$