Solutions to Tutorial 7

Question 1 (a). The limiting function $F(x) = \lim_{n\to\infty} \dfrac{n^2}{n^2+x^2} = \lim_{n\to\infty} \dfrac{1}{1+\frac{x^2}{n^2}} = 1$.

$$0 \le T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left|\frac{n^2}{n^2+x^2} - 1\right| = \sup_{0\le x\le 1} \frac{x^2}{n^2+x^2} \le \frac{1}{n^2}.$$

Since $\lim_{n\to\infty} \dfrac{1}{n^2} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{F_n\}$ converges uniformly on $[0,1]$. □

Question 1 (b). The limiting function $F(x) = \lim_{n\to\infty} x^n(1-x) = 0$ for $0 \le x \le 1$. Observe

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} x^n(1-x).$$

Let $g(x) = x^n(1-x)$. Then $g'(x) = nx^{n-1}(1-x) - x^n = x^{n-1}[n-(n+1)x]$. From $g'(x) = 0$, we have $x = 0$ or $\frac{n}{n+1}$. Since $g'(x) \ge 0$ for $0 \le x \le \dfrac{n}{n+1}$ and $g'(x) \le 0$ for $\dfrac{n}{n+1} \le x \le 1$, $\sup_{0\le x\le 1} g(x) = \max\{g(x) | 0 \le x \le 1\} = \left(\dfrac{n}{n+1}\right)^n \left(1 - \dfrac{n}{n+1}\right)$ and so

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.$$

Thus $\{F_n\}$ converges uniformly on $[0,1]$. □

Question 1 (c). The limiting function $f(x) = \lim_{n\to\infty} \dfrac{n\ln x}{x^n} = 0$ for $1 \le x < \infty$. Observe that

$$T_n = \sup_{x\ge 1} |f_n(x) - f(x)| = \sup_{x\ge 1} \left|\frac{n\ln x}{x^n} - 0\right| = \sup_{x\ge 1} \frac{n\ln x}{x^n}.$$

Let $g(x) = \dfrac{n\ln x}{x^n}$. From

$$g'(x) = \left(n\ln x \cdot x^{-n}\right)' = n\frac{1}{x}x^{-n} - n^2\ln x \cdot x^{-n-1} = \frac{n - n^2\ln x}{x^{n+1}} = 0,$$

we have $x = e^{\frac{1}{n}}$. Since $g'(x) \ge 0$ for $1 \le x \le e^{\frac{1}{n}}$ and $g'(x) \le 0$ for $x \ge e^{\frac{1}{n}}$, we have

$$T_n = \sup_{x\ge 1} g(x) = \max\{g(x) | x \ge 1\} = \frac{n\ln e^{\frac{1}{n}}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{1}{e}$$

and so $\lim_{n\to\infty} T_n = \dfrac{1}{e} \ne 0$. Thus $\{f_n\}$ does not converge uniformly on $[1, +\infty)$. □

Question 1 (d). The limiting function $f(x) = \lim_{n\to\infty} \frac{n \ln x \cos nx}{x^n} = 0$ for $x \geq 4$. Observe that

$$0 \leq T_n = \sup_{x\geq 4} |f_n(x) - f(x)| = \sup_{x\geq 4} \frac{n \ln x \cdot |\cos nx|}{x^n} \leq \sup_{x\geq 4} \frac{n \ln x}{x^n} = g(4) = \frac{n \ln 4}{4^n},$$

where $g(x) = \frac{n \ln x}{x^n}$ is monotone decreasing on $[4, +\infty)$. Since $\lim_{n\to\infty} \frac{n \ln 4}{4^n} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{f_n\}$ converges uniformly on $[4, +\infty)$. □

Question 1 (e). The limiting function $F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{n^2}{n^2 + x^2} = 1$. Observe that

$$T_n = \sup_{x\geq 0} |F_n(x) - F(x)| = \sup_{x\geq 0} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{x\geq 0} \frac{x^2}{n^2 + x^2}.$$

Let $g(x) = \frac{x^2}{n^2 + x^2}$. Since

$$g'(x) = \frac{2x(n^2 + x^2) - x^2 \cdot 2x}{(n^2 + x^2)^2} = \frac{2xn^2}{(n^2 + x^2)^2} \geq 0$$

for $x \geq 0$, the function $g(x)$ is monotone increasing and so

$$T_n = \sup_{x\geq 0} g(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^2}{n^2 + x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

Thus $\lim_{n\to\infty} T_n = 1 \neq 0$ and so $\{F_n(x)\}$ does not converge uniformly on $[0, +\infty)$. □

Question 2 (i). Since

$$\left| \frac{\cos nx}{n^2 + x^2} \right| \leq \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M-test. □

Question 2 (ii). Since

$$\left| \frac{1}{1 + n^3 x^2} \right| \leq \frac{1}{1 + 2^2 n^3} \leq \frac{1}{n^3}$$

for $x \geq 2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{1}{1 + n^3 x^2}$ converges uniformly on $[2, +\infty)$ by the Weierstrass M-test. □

Question 2 (iii). Let $f_n(x) = \dfrac{xe^{-x}}{n^2}$ for $x \in (0, +\infty)$. Then

$$f_n'(x) = \frac{1}{n^2}\left(e^{-nx} - xne^{-nx}\right) = \frac{(1-nx)e^{-nx}}{n^2}.$$

Thus $f_n(x)$ is monotone increasing for $0 \le x \le \dfrac{1}{n}$ and monotone decreasing for $x \ge \dfrac{1}{n}$. It follows that

$$|f_n(x)| \le f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}e^{-n\cdot\frac{1}{n}}}{n^2} = \frac{e^{-1}}{n^3}$$

for $x \in (0, +\infty)$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-1}}{n^3} = e^{-1}\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent by the p-series, the series of functions $\displaystyle\sum_{n=1}^{\infty}\frac{xe^{-nx}}{n^2}$ converges uniformly on $(0, +\infty)$ by the Weierstrass M-test. □

Question 3. Since the series of functions $\displaystyle\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on I, by the Cauchy Criterion, for any $\epsilon > 0$, there exists N such that

$$\sum_{k=n+1}^{m} g_k(x) = \left|\sum_{k=n+1}^{m} g_k(x)\right| < \epsilon$$

for all $x \in I$ and $m > n > N$. Because $g_k(x) \ge |f_k(x)| \ge 0$, for all $x \in I$ and $n > m > N$, we have

$$\left|\sum_{k=n+1}^{m} f_k(x)\right| \le \sum_{k=m+1}^{n} |f_k(x)| \le \sum_{k=n+1}^{m} g_k(x) < \epsilon.$$

Thus, by the Cauchy Criterion, the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly. □

Question 4. Answer: NO. Suppose that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^x}$ converges uniformly on $(0, +\infty)$. By the Cauchy Criterion, for given $\epsilon = \dfrac{1}{2}$ there exists N such that

$$\left|\sum_{k=n+1}^{m} (-1)^{n+1}\frac{1}{n^x}\right| < \frac{1}{2}$$

for all $n, m > N$ and $x \in (0, +\infty)$. In particular, let $n = N + 1$ and $m = N + 2$,

$$\frac{1}{(N+2)^x} = \left| \sum_{k=N+2}^{N+2} (-1)^{n+1} \frac{1}{n^x} \right| < \frac{1}{2}$$

for all $x \in (0, +\infty)$. This contradicts to that

$$\frac{1}{(N+2)^x} \geq \frac{1}{2}$$

when $0 < x \leq \dfrac{\ln 2}{\ln(N+2)}$. □