

Solutions to Tutorial 8

Question 1 (i). Let $F_n(x) = \frac{n + e^x}{n + x^2}$. Then the limiting function

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \frac{n + e^x}{n + x^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{e^x}{n}}{1 + \frac{x^2}{n}} = 1$$

and

$$\begin{aligned} 0 \leq T_n &= \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} \left| \frac{n + e^x}{n + x^2} - 1 \right| \\ &= \sup_{0 \leq x \leq 1} \frac{|e^x - x^2|}{n^2 + x^2} \leq \sup_{0 \leq x \leq 1} \frac{e^x + x^2}{n^2 + x^2} \leq \frac{e + 1}{n^2} \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{e + 1}{n^2} = \lim_{n \rightarrow \infty} 0 = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze Theorem. Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{n + e^x}{n + x^2} dx = \int_0^1 \lim_{n \rightarrow \infty} F_n(x) dx = \int_0^1 1 dx = 1.$$

□

Question 1 (ii). Let $F_n(x) = \left(\frac{x^2 + 1}{8}\right)^n \sin nx$. Then

$$0 \leq |F_n(x)| = \left| \left(\frac{x^2 + 1}{8}\right)^n \sin nx \right| \leq \left(\frac{2^2 + 1}{8}\right)^n = \left(\frac{5}{8}\right)^n$$

for $1 \leq x \leq 2$. Since $\lim_{n \rightarrow \infty} \left(\frac{5}{8}\right)^n = \lim_{n \rightarrow \infty} 0 = 0$, the limiting function $F(x) = 0$ for $1 \leq x \leq 2$ and

$$0 \leq T_n = \sup_{1 \leq x \leq 2} |F_n(x) - F(x)| = \sup_{1 \leq x \leq 2} |F_n(x)| \leq \left(\frac{5}{8}\right)^n$$

Since $\lim_{n \rightarrow \infty} \left(\frac{5}{8}\right)^n = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze Theorem and so $\{F_n\}$ converges uniformly to $F(x)$. Thus

$$\lim_{n \rightarrow \infty} \int_1^2 \left(\frac{x^2 + 1}{8}\right)^n \sin nx dx = \int_1^2 \lim_{n \rightarrow \infty} F_n(x) dx = \int_1^2 0 dx = 0.$$

□

Question 2. Let $f_k(x) = \frac{(-1)^k x^k}{1+x^{2k}}$ for $0 < x < \frac{2}{3}$. Since

$$|f_k(x)| = \left| \frac{(-1)^k x^k}{1+x^{2k}} \right| \leq \frac{\left(\frac{2}{3}\right)^k}{1} = \left(\frac{2}{3}\right)^k$$

for $0 < x < \frac{2}{3}$ and the series $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges by the geometric series, the series

of functions $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ converges uniformly on $\left(0, \frac{2}{3}\right)$ by the Weierstrass M -test.

Thus the function $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ is continuous on the interval $\left(0, \frac{2}{3}\right)$. \square

Question 3. Let $f_n(x) = \frac{x^n(1-x^2)}{\sqrt{1+x}}$ for $0 \leq x \leq \frac{1}{2}$. Since

$$|f_n(x)| = \frac{|x|^n |1-x^2|}{\sqrt{1+x}} \leq \frac{\left(\frac{1}{2}\right)^n \cdot 1}{1} = \left(\frac{1}{2}\right)^n$$

for $0 \leq x \leq \frac{1}{2}$ and the series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent by the geometric series, the

series of functions $\sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}}$ converges uniformly on $\left[0, \frac{1}{2}\right]$ by the M -test. Note

that each $f_n(x)$ is Riemann integrable on $\left[0, \frac{1}{2}\right]$. We have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} \frac{x^n(1-x^2)}{\sqrt{1+x}} dx &= \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}} dx = \int_0^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} x^n \right) \cdot \frac{1-x^2}{\sqrt{1+x}} dx \\ &= \int_0^{\frac{1}{2}} \frac{1}{1-x} \cdot \frac{1-x^2}{\sqrt{1+x}} dx = \int_0^{\frac{1}{2}} \sqrt{1+x} dx = \frac{2}{3} (1+x)^{\frac{3}{2}} \Big|_0^{\frac{1}{2}} = \frac{2}{3} \left[\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1 \right] = \sqrt{\frac{3}{2}} - \frac{2}{3}. \end{aligned}$$

\square

Question 4 (i). Since

$$|a_k \sin kx| \leq |a_k|$$

for $x \in (-\infty, +\infty)$ and $\sum_{k=1}^{\infty} |a_k|$ is convergent, the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $(-\infty, +\infty)$ by the M -test. \square

Question 4 (ii). By (i), the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $[0, 2\pi]$. Since each $a_k \sin kx$ is Riemann integrable, we have

$$\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \sin kx dx = \sum_{k=1}^{\infty} \int_0^{2\pi} a_k \sin kx dx = \sum_{k=1}^{\infty} 0 = 0.$$

□

Question 5. From $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x , we have

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}.$$

Since

$$\left| \frac{(-1)^n x^{3n}}{n!} \right| \leq \frac{1}{n!}$$

for $0 \leq x \leq 1$ and the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent by the ratio test, the series of

functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$ converges uniformly to e^{-x^3} on $[0, 1]$ by the M -test. Note that each $\frac{(-1)^n x^{3n}}{n!}$ is Riemann integrable. Thus

$$\int_0^1 e^{-x^3} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(3n+1)}.$$

Let $a_n = \frac{1}{n!(3n+1)}$. Then the sequence $\{a_n\}$ is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. By applying the alternating test estimation, from $a_{n+1} < 0.001$ or $(n+1)!(3n+4) \geq 1000$, we have $n \geq 4$ and so

$$\int_0^1 e^{-x^3} dx \approx 1 - \frac{1}{1! \cdot 4} + \frac{1}{2! \cdot 7} - \frac{1}{3! \cdot 10} + \frac{1}{4! \cdot 13} = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}$$

with error less than 0.001. □