Solutions to Tutorial 9

Question 1. Let $f_n(x) = \dfrac{\cos^n x}{n^3}$. Then

(1). Each $f_n'(x) = \dfrac{-\cos^{n-1} x \sin x}{n^2}$ is continuous on $(-\infty, +\infty)$.

(2). The series of functions $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges on $(-\infty, +\infty)$ by the comparison test because $\left|\dfrac{\cos^n x}{n^3}\right| \leq \dfrac{1}{n^3}$ and the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ is convergent by the p-series. So it converges pointwise on $(-\infty, +\infty)$.

(3). The series of functions $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M-test because $\left|\dfrac{-\cos^{n-1} x \sin x}{n^2}\right| \leq \dfrac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges by the p-series.

Thus the function $f(x) = \sum_{n=1}^{\infty} \dfrac{\cos^n x}{n^3}$ is differentiable $(-\infty, +\infty)$ by Corollary 5.4 and Remark 5.2 to Theorem 5.1.

□

Question 2 (a). Let b be any given positive number. Let $f_n(t) = \dfrac{t^n}{n!}$ and let

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Then

1) $f_n'(t)$ exists and is continuous on $[-b, b]$;
2) the series of functions $\sum_{n=0}^{\infty} f_n(t)$ converges pointwise on $[-b, b]$ and
3) the series of functions $\sum_{n=0}^{\infty} f_n'(t)$ converges uniformly on $[-b, b]$, by the Weierstrass M-test, because

$$|f_n'(t)| = \left|\frac{t^{n-1}}{(n-1)!}\right| \leq \frac{b^{n-1}}{(n-1)!}$$

and $\sum_{n=1}^{\infty} \dfrac{b^{n-1}}{(n-1)!}$ converges.

Thus

$$f'(t) = \sum_{n=0}^{\infty} \left(\frac{t^n}{n!}\right)' = 0 + 1 + t + \frac{t^2}{2!} + \cdots = f(t)$$

for any $t \in [-b, b]$. Since b is any given positive number, $f'(t) = f(t)$ for all $t \in (-\infty, +\infty)$ and so $y = f(x)$ is a solution of $y' = y$. □

Question 2 (b).

$$\frac{dy}{dt} = y \quad \Rightarrow \quad \int \frac{dy}{y} = \int dt$$

$$\Rightarrow \ln|y| = t + k \quad \Rightarrow \quad y = Ce^t,$$

where $C = \pm e^k$. Let

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

From (i), $f(t) = Ce^t$ for certain constant C. Let $t = 0$, we have

$$C \cdot e^0 = f(0) = 1 + 0 + 0 + \cdots = 1$$

and so $f(t) = e^t$. Hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in (-\infty, +\infty)$. □

Question 3 (i).

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup \sqrt[n]{\left(1 + \frac{3}{n}\right)^{n^2}}} = \frac{1}{\lim\limits_{n\to\infty} \left(1 + \frac{3}{n}\right)^n} = \frac{1}{e^3}.$$

□

Question 3 (ii).

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{|-3|^{n+1} \cdot n!}{(n+1)! \cdot |-3|^n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3}{n+1}} = +\infty.$$

□

Question 3 (iii). Since $a_n = \left(\frac{1}{5}\right)^n$ if n is odd and $\left(\frac{1}{6}\right)^n$ if n is even, $\sqrt[n]{|a_n|} = \frac{1}{5}$ if n is odd and $\frac{1}{6}$ if n is even. Thus

$$b_n = \sup_{k \geq n} \sqrt[n]{|a_n|} = \frac{1}{5}$$

and so $\limsup \sqrt[n]{|a_n|} = \lim_{n\to\infty} b_n = \frac{1}{5}$. It follows that

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 5.$$

□

Question 3 (iv). Observe that

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{3^n}{n^2} \cdot (x - \frac{2}{3})^n.$$

Thus

$$R = \frac{1}{\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{3^{n+1}\cdot n^2}{(n+1)^2\cdot 3^n}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{3}{\left(1+\frac{1}{n}\right)^2}} = \frac{1}{3}.$$

□

Question 4.

$$\sqrt[3]{9} = (8+1)^{\frac{1}{3}} = 2\left(1+\frac{1}{8}\right)^{\frac{1}{3}} = 2\sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \frac{1}{8^k}$$

$$= 2 + \sum_{k=1}^{\infty} \frac{\frac{1}{3}\cdot(\frac{1}{3}-1)\cdots(\frac{1}{3}-k+1)}{k!\cdot 2^{3k-1}}$$

$$= 2 + \frac{1}{12} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}}$$

Let $a_k = \dfrac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}}$ for $k \geq 2$. Then

1) $a_k > 0$, that is, the series is alternating.
2) Since

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3})}{(k+1)!\cdot 2^{3k+2}} \cdot \frac{k!\cdot 2^{3k-1}}{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}$$

$$= \frac{k-\frac{1}{3}}{(k+1)\cdot 8} \leq 1,$$

$a_{k+1} \leq a_k$ or $\{a_k\}$ is monotone deceasing.

3) Observe that

$$0 \leq a_k = \frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}} < \frac{\frac{1}{3}\cdot 1\cdot 2\cdots(k-1)}{k!\cdot 2^{3k-1}} = \frac{1}{3\cdot k\cdot 2^{3k-1}}$$

for $k \geq 2$. Since $\lim_{k\to\infty} \dfrac{1}{3\cdot k\cdot 2^{3k-1}} = 0$, $\lim_{k\to\infty} a_k = 0$ by the Squeeze theorem.

Thus we can apply the alternating series estimation. From

$$a_{k+1} = \frac{\frac{1}{3} \cdot (1 - \frac{1}{3}) \cdots (k - \frac{1}{3})}{(k+1)! \cdot 2^{3k+2}} < 10^{-4},$$

we have $k \geq 3$. Thus

$$\sqrt[3]{9} \approx 2 + \frac{1}{12} - \frac{1}{288} + \frac{5}{20746} = 2.0798853352131236495063128009844 7$$

with an error of magnitude less than 10^{-4}. $\square$