NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA2108 ADVANCED CALCULUS II

November 2003 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\arctan\left(\sin\dfrac{(\ln n)^2}{\sqrt{n}}\right)+3+\dfrac{n^5}{1.1^n}\right\}$.

(b) $\left\{\dfrac{(\ln n)^3+8n!+6^n\cdot n^{100}}{n!+n^9}\right\}$.

(c) $\left\{\left(1+2n^2\right)^{1/n}\right\}$.

(d) $\left\{\sqrt[n]{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}}\right\}$.

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-6n}{n^3+3n+1}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n^2+1}{1+8n^2}\right)^n$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$.

(d) $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{n}{1+n^2}\right)$.

Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n2^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}x^n$.

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converges uniformly on the indicated intervals. Justify your answers.

$$F_n(x) = 2x(1-x)^n\sqrt{n}, \quad x \in [0,1].$$

(b) Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{\sqrt{k^3+x^3}}, \quad x \in [0,\infty).$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+2x}} \quad x \in [0,\infty).$

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\lim_{n\to\infty} \int_0^1 \left(\frac{x^2+x+2}{5}\right)^n \cos(nx^2)\, dx$. Justify your answer.

(b) Find the **interval of convergence** of the power series

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{1+n\ln n}.$$

Justify your answer.

(c) Let $\{a_n\}$ be a **convergent** sequence. Define

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Does the sequence $\{b_n\}$ converge? Justify your answer.

Question 6 [20 marks]

(a) Using any applicable method, find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt[4]{16-2x}}$ at $x_0 = 0$, and determine its radius of convergence.

(b) Suppose that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges on an interval I. Prove that the sequence of functions $\{f_n(x)\}$ converges uniformly to 0 on I.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Show that

$$\overline{\lim_{n\to\infty}}\,(a_n + b_n) \le \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

Question 7 [20 marks]

(a) Let $f(x) = x^3 \cdot \sin(x^8)$. Find $f^{(48)}(0)$.

(b) Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1 \qquad a_{n+1} = 1 + \frac{a_n}{a_n + 1} \quad n \geq 1.$$

Prove that $\{a_n\}$ is convergent, and find its limit.

(c) Show that the series

$$1 + \sum_{n=1}^{\infty} \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

converges for any real number $a > -1$.