

Take-home Exam 1

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam already.

Question 1. [2 points, 1 for each part]

Prove the following limits by using $\epsilon - N$ definition

- i) $\lim_{n \rightarrow \infty} \frac{3n+8}{2n+9} = \frac{3}{2}$.
- ii) $\lim_{n \rightarrow \infty} \frac{(-1)^n n}{n^2+1} = 0$.

Question 2. [5 points, 1 for each part]

For each of the following sequences, either find the limit or show that the limit does not exist.

- (a) $\left\{ \left(\sqrt{n^2+n} - n \right) \right\}$.
- (b) $\left\{ \left(2^n + 3^n \right)^{\frac{1}{n}} \right\}$.
- (c) $\left\{ \sqrt[4]{ \frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n} } \right\}$.
- (d) $\left\{ \left(\frac{3n}{3n-1} \right)^{2n+\sqrt{n}} \right\}$.
- (e) $\left\{ \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right\}$.

Question 3. [3 points, 1 for each part]

- (a) If $\{a_n\}$ is convergent, show that $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$.
- (b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1+a_n)$ for $n \geq 1$. Assume that $\{a_n\}$ is convergent, find its limit.
(Hint: there will be two possible values for $\lim_{n \rightarrow \infty} a_n$. For rejecting one of them, show by induction that $a_n > 0$.)
- (c) Find the limit of the sequence

$$\left\{ \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \dots \right\}.$$

(Hint: This sequence is given recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. You can show by induction that $0 \leq a_n \leq 2$. Then prove that $\{a_n\}$ is monotone increasing. By applying monotone convergence theorem, conclude that $\lim_{n \rightarrow \infty} a_n$ exists. There will be two possible values for $\lim_{n \rightarrow \infty} a_n$ and you have to decide to reject one of them.)