2003/2004 Semester I MA2108 Advanced Calculus II

Take-home Exam 5

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

Question 1. [6 points, 1 for each part] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \quad x \in [0, \infty).$

(b) $\displaystyle\sum_{k=1}^{\infty} e^{-kx} x^k, \quad x \in [0, \infty).$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}, \quad x \in [0, \infty).$

Hint: Let $S(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$ and let $S_n(x) = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+x}$. By using alternating series estimation, show that

$$T_n = \sup_{x \geq 0} |S_n(x) - S(x)| \leq \sup_{x \geq 0} \frac{1}{n+1+x} \leq \frac{1}{n+1}.$$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \quad x \in [-1, 1].$

(e) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right), \quad x \in [0, 1].$

(Hint: Find the partial sums $S_n(x) = \displaystyle\sum_{k=0}^{n} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2} \right)$ and then show that $S(x) = \displaystyle\lim_{n \to \infty} S_n(x)$ is not continuous while each $S_n(x)$. From this, conclude that the series of functions does not converge uniformly.)

(f) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{x}{2} \right)^k, \quad x \in (-2, 2).$

(Hint: Try T-test. Show that $T_n = \displaystyle\sup_{-2<x<2} \left| \sum_{k=n+1}^{\infty} \left(\frac{x}{2} \right)^k \right| = +\infty$ by letting $x \to 2^-$.)

Question 2 [4 points, 1 for each part] Find the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k.$

(b) $\displaystyle\sum_{k=1}^{\infty} \left(1 - \frac{1}{k} \right)^{k^2} (x-1)^k.$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k$

(d) $\displaystyle\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$.

(*Hint.* For part (d), you can use the root test to find the interval for which the series converges absolutely.)