PROBLEM-BASED LEARNING FOR ADVANCED CALCULUS II

WU JIE

Proposed Problem: Computation of Pi

Step (1). Recall the formula

$$\frac{\pi}{6} = \arcsin\frac{1}{2}.$$

Step (2). Recall the integral formula that

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^x \left(1-t^2\right)^{-\frac{1}{2}}\,dt.$$

Step (3). Recall the formula

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Generalization of (3): Binomial Series

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots$$

holds for any real number α, where the binomial number

$$\binom{\alpha}{k} = \frac{\alpha\cdot(\alpha-1)\cdot\cdots\cdot(\alpha-k+1)}{k!}.$$

- We will prove that this formula holds when $|x| < 1$.
- If α is non-negative integer, the binomial number $\binom{\alpha}{k}$ is never zero and so the right hand side of the above formula is an **infinite summation**, which is called **(infinite) series** that we are going to study in this module.

Step (4). From the above, by letting $\alpha = -\frac{1}{2}$ and replacing x to be $-t^2$, obtain the formula

$$(1-t^2)^{-\frac{1}{2}} = 1+\binom{-\frac{1}{2}}{1}(-t^2)+\binom{-\frac{1}{2}}{2}(-t^2)^2+\binom{-\frac{1}{2}}{3}(-t^2)^3+\cdots.$$

Step (5). From (4),

$$\frac{\pi}{6} = \int_0^{\frac{1}{2}} (1-t^2)^{-\frac{1}{2}}\, dt$$

$$= \int_0^{\frac{1}{2}} \left[1+\binom{-\frac{1}{2}}{1}(-t^2)+\binom{-\frac{1}{2}}{2}(-t^2)^2+\binom{-\frac{1}{2}}{3}(-t^2)^3+\cdots\right] dt$$

$$\overset{??}{=\!=\!=} \int_0^{\frac{1}{2}} 1\, dt + \int_0^{\frac{1}{2}} \binom{-\frac{1}{2}}{1}(-t^2)\, dt + \int_0^{\frac{1}{2}} \binom{-\frac{1}{2}}{2}(-t^2)^2\, dt + \int_0^{\frac{1}{2}} \binom{-\frac{1}{2}}{3}(-t^2)^3\, dt + \cdots,$$

where the above question mark means that it needs proof, but at the moment we temporarily assume that this is true to see what's going on.

By working out each of the above integrals, there is a (conjectured) formula

$$\frac{\pi}{6} = \frac{1}{2} - \binom{-\frac{1}{2}}{1}\frac{1}{3\cdot 2^3} + \binom{-\frac{1}{2}}{2}\frac{1}{5\cdot 2^5} - \binom{-\frac{1}{2}}{3}\frac{1}{7\cdot 2^7} + \cdots$$

$$= \frac{1}{2}+\frac{1}{3\cdot 2^4}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{2\cdot 1}\frac{1}{5\cdot 2^5}-\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{3\cdot 2\cdot 1}\frac{1}{7\cdot 2^7}+\cdots$$

By only taking the first term, we compute

$$\pi \approx 6 \cdot \frac{1}{2} = 3.$$

By taking the first two terms,

$$\pi \approx 6 \cdot \left(\frac{1}{2} + \frac{1}{3 \cdot 2^4}\right) = 3 + \frac{1}{8} = 3.125,$$

which is more accurate.

By taking the first three terms,

$$\pi \approx 6 \cdot \left(\frac{1}{2} + \frac{1}{3 \cdot 2^4} + \frac{3}{5 \cdot 2^8}\right) = 3 + \frac{1}{8} + \frac{9}{640} = 3.1390625,$$

which is even more accurate.

Step (6). Let's ask the computer to check this formula. By adding up first 10-terms, Maple answers that we get 8 digits correct for π.

In the last lecture, we will fill-in the proofs for the above computation. Arising from this problem, we need to learn:

1) How can we know an infinite series adding up to a finite number, namely **Convergence/Divergence**?
2) How to obtain the binomial series? Can we write down other functions like $\sin x$, $\cos x$, e^x as an infinite summation of powers of x? These will be answered by studying **sequences and series of functions**, **power series**, **Taylor series** and etc.
3) For an infinite summation of functions, can we do integrals and derivatives term-by-term? The answer is that **sometimes is yes and sometimes is no.** One needs a condition so-called **uniform convergence**.