MA2108 **Midterm** **5 October 2004**

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Tuesday 3-4 (ST1)	Tuesday 4-5 (ST2)	Wednesday 11-12 (ST3)
Wednesday 12-1 (ST4)	Thursday 11-12 (ST5)	Thursday 12-1 (ST6)
Friday 12-1 (ST7)	Friday 1-2 (ST8)	Saturday 11-12 (ST9)
Saturday 12-1 (ST10)	Friday 4-6 (STL)	Wednesday 2-4 (S-version)

Matriculation number:________________________

Name:________________________

Signature:________________________

INSTRUCTIONS TO CANDIDATES

1. This test contains a total of **TWELVE (12)** Questions.
2. Answer **ALL** questions.
3. The examination carries a total of 50 marks.
4. Candidates may use a help-sheet, up to half of A4 size (both sided).

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Problem #	Your Grades
1 (4 points)	
2 (4 points)	
3 (4 points)	
4 (4 points)	
5 (4 points)	
6 (4 points)	
7 (4 points)	
8 (4 points)	
9 (5 points)	
10 (4 points)	
11 (4 points)	
12 (5 points)	
total (50 points)	

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Determine the limit of the following sequences. Justify your answer.

1. [4 marks] $\left\{ \dfrac{n^3 - 100n^2 + n - 58}{2n^3 + 65n + 69} \right\}$.

Solution.

$$\begin{aligned} &\lim_{n\to\infty} \frac{n^3 - 100n^2 + n - 58}{2n^3 + 65n + 69} \\ = &\lim_{n\to\infty} \frac{1 - 100/n + 1/n^2 - 58/n^3}{2 + 65/n^2 + 69/n^3} = 1/2. \end{aligned}$$

□

2. [4 marks] $\left\{ \left(\sqrt{4 - \dfrac{1}{n}} - 2 \right) n \right\}$.

Solution.

$$\begin{aligned} &\lim_{n\to\infty} \left(\sqrt{4 - \frac{1}{n}} - 2 \right) n \\ = &\lim_{n\to\infty} \frac{\left(\left(4 - \frac{1}{n}\right) - 2^2 \right) n}{\left(\sqrt{4 - \frac{1}{n}} + 2 \right)} \\ = &-\frac{1}{\sqrt{4} + 2} = -\frac{1}{4}. \end{aligned}$$

□

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Determine the limit of the following sequences. Justify your answer.

3. [4 marks] $\left\{\dfrac{\sin(n^2)}{\sqrt{n}}\right\}$.

Solution.

$$-\frac{1}{\sqrt{n}} \leq \frac{\sin(n^2)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $\lim\limits_{n\to\infty} \frac{\sin(n^2)}{\sqrt{n}} = 0$ by the Squeeze Theorem. □

4. [4 marks] $\left\{\dfrac{\sqrt[n]{(2n)!}}{n^2}\right\}$.

Solution. Let $a_n = \dfrac{(2n)!}{n^{2n}}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt[n]{(2n)!}}{n^2} &= \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(2n+2)! \cdot n^{2n}}{(n+1)^{2n+2} \cdot (2n)!} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \left[\left(1 + \frac{1}{n}\right)^n\right]^{-2} \\
&= 4e^{-2}.
\end{aligned}$$

□

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Determine convergence or divergence of the following series. Justify your answer.

5. [4 marks] $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+18}}{n^2+4n+3}$.

Solution. Convergence by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Let $a_n = \frac{\sqrt{n+18}}{n^2+4n+3}$ and let $b_n = \frac{1}{n^{3/2}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+18}}{n^2+4n+3} \cdot n^{3/2} = \lim_{n\to\infty} \frac{\sqrt{1+18/n}}{1+4/n+3/n^2} = 1.$$

□

6. [4 marks] $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1}$.

Solution. The series is divergent by divergence test.

Let $a_n = (-1)^n \frac{n+1}{2n+1}$. Since

$$\lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{1+1/n}{2+1/n} = \frac{1}{2},$$

the limit of a_n does not exists and so the series diverges by divergence test. □

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Determine convergence or divergence of the following series. Justify your answer.

7. [4 marks] $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(n!)}{n^n}$.

Solution. Convergence by ratio test. Let $a_n = \frac{2^n(n!)}{n^n}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{2^{n+1}\cdot(n+1)!n^n}{(n+1)^{n+1}\cdot 2^n\cdot n!} \\ &= \lim_{n\to\infty} \frac{2}{(1+1/n)^n} = 2/e < 1.\end{aligned}$$

□

8. [4 marks] $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^{3/2}}$.

Solution. Convergence by integral test.

Let $f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^{3/2}}$. Then $f(x)$ is positive and monotone decreasing on $[3,\infty)$. Since the integral

$$\begin{aligned}\int_3^{\infty} f(x)\,dx &= \int_3^{\infty} \frac{1}{x(\ln x)[\ln(\ln x)]^{3/2}}\,dx \\ &\overset{y=\ln x}{=\!=} \int_{\ln 3}^{\infty} \frac{1}{y(\ln y)^{3/2}}\,dy \\ &\overset{z=\ln y}{=\!=} \int_{\ln\ln 3}^{\infty} \frac{1}{z^{3/2}}\,dz \\ &= \frac{1}{-3/2+1} z^{-1/2}\Big|_{\ln\ln 3}^{\infty} = \frac{2}{\sqrt{\ln\ln 3}}\end{aligned}$$

is convergent, the series converges. □

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Determine the absolute convergence, conditional convergence or divergence of the following series. Justify your answer.

9. [5 marks] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+6)}{2n^2+1}$.

Solution. Conditional convergence.

The series converges by the alternating series test:

$$\lim_{n\to\infty} \frac{(n+6)}{2n^2+1} = \lim_{n\to\infty} \frac{1+6/)}{2n+1/n} = 0.$$

To see that $\{\frac{(n+6)}{2n^2+1}\}$ is monotone decreasing. Let $f(x) = \frac{x+6}{2x^2+1}$, then

$$f'(x) = \frac{2x^2+1-(x+6)(4x)}{(2x^2+1)^2} = \frac{-2x^2-24x+1}{2x^2+1} < 0$$

on $[1,\infty)$. Thus the function $f(x)$ is monotone decreasing on $[1,\infty)$ and in particular, the sequence $\{\frac{(n+6)}{2n^2+1}\}$ is monotone decreasing.

The series $\sum\limits_{n=1}^{\infty} \frac{(n+6)}{2n^2+1}$ is divergent by limit comparison test with $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. Let $a_n = \frac{(n+6)}{2n^2+1}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+6)}{2n^2+1} \cdot n = \lim_{n\to\infty} \frac{1+6/n}{2+1/n^2} = \frac{1}{2}.$$

□

10. [4 marks] $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2n}{2n+1}\right)^{n(n+1)}$.

Solution. Absolute convergence by the root test. Let $a_n = (-1)^{n+1}\left(\frac{2n}{2n+1}\right)^{n(n+1)}$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1/2}{n}\right)^{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1/2}{n}\right)^n \cdot \left(1+\frac{1/2}{n}\right)} = \frac{1}{\sqrt{e}} < 1.$$

□

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11. [4 marks] Find limit inferior and limit superior of the sequences:

$$\left\{\sin\left(\frac{n\pi}{2}\right)+\cos(n\pi)\right\}.$$

Justify your answer.

Solution. The sequence

$$\begin{aligned}&\left\{\sin\left(\frac{n\pi}{2}\right)+\cos(n\pi)\right\}\\ &=\{1-1,0+1,-1-1,0+1,1-1,0+1,-1-1,0+1,\ldots\}\\ &=\{0,1,-2,1,0,1,-2,1,0,1,-2,1,\ldots\}.\end{aligned}$$

The subsequential limits are $0,1,-2$.

Thus $\overline{\lim_{n\to\infty}}\sin\left(\frac{n\pi}{2}\right)+\cos(n\pi)=1$ and $\underline{\lim_{n\to\infty}}\sin\left(\frac{n\pi}{2}\right)+\cos(n\pi)=-2$. □

12. [5 marks] Let $\{a_n\}$ be the sequence defined by

$$a_1=3 \qquad a_{n+1}=1+\sqrt{a_n-1}.$$

Does the sequence $\{a_n\}$ converges? If so, find its limit. Justify your answer.

Solution. First $\{a_n\}$ is monotone decreasing. $a_1=3$ $a_2=1+\sqrt{3-1}\le a_1$. Suppose that $a_n\le a_{n-1}$. Then

$$\begin{aligned}a_{n+1}-a_n&=(1+\sqrt{a_n-1})-(1+\sqrt{a_{n-1}-1})\\ &=\frac{a_n-a_{n-1}}{(1+\sqrt{a_n-1})+(1+\sqrt{a_{n-1}-1})}\le 0,\end{aligned}$$

that is $a_{n+1}\le a_n$. Hence $\{a_n\}$ is monotone decreasing by induction.

Second $\{a_n\}$ is bounded below by 2. $a_1=3\ge 2$. Suppose that $a_n\ge 2$. Then $a_{n+1}=1+\sqrt{a_n-1}\ge 1+\sqrt{2-1}=2$. Thus $\{a_n\}$ is bounded below by 2 by induction.

By monotone convergence theorem, $\{a_n\}$ converges. Let $A=\lim_{n\to\infty}a_n$. Then

$$A=1+\sqrt{A-1}\Longrightarrow A=1,2.$$

Since $\{a_n\}$ is bounded below by 2, $A\ge 2$. Thus $A=1$ is rejected and hence $A=2$.

□