NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2004-2005

MA2108/MA2108S ADVANCED CALCULUS II

November 2004 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A
Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\dfrac{1+n-3n^2}{3-2n+n^2}\right\}$.

(b) $\left\{\dfrac{\sin(n^2+1)}{n}\right\}$.

(c) $\left\{(2^n+3^n)^{1/n}\right\}$.

(d) $\left\{\left(1-\dfrac{1}{n^2}\right)^{n^2+1}\right\}$.

Solution. (a).

$$\lim_{n\to\infty}\frac{1+n-3n^2}{3-2n+n^2}=\lim_{n\to\infty}\frac{1/n^2+1/n-3}{3/n^2-2/n+1}=-3$$

(b).

$$-\frac{1}{n}\le\frac{\sin(n^2+1)}{n}\le\frac{1}{n}$$

Since $\lim_{n\to\infty}\frac{1}{n}=0$, $\lim_{n\to\infty}\frac{\sin(n^2+1)}{n}=0$ by the Squeeze Theorem.

(c).

$$\lim_{n\to\infty}(2^n+3^n)^{1/n}=\lim_{n\to\infty}3\left((2/3)^n+1\right)^{1/n}=3\cdot(0+1)^0=3.$$

Or $3\le(2^n+3^n)^{1/n}\le(2\cdot 3^n)^{1/n}=3\cdot 2^{1/n}$. Since $\lim_{n\to\infty}3\cdot 2^{1/n}=3$, $\lim_{n\to\infty}(2^n+3^n)^{1/n}=3$ by the Squeeze Theorem.

(d).

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{n^2+1}=\lim_{n\to\infty}\left[\left(1-\frac{1}{n^2}\right)^{n^2}\right]^{1+1/n^2}=\left[e^{-1}\right]^{1+0}=e^{-1}.$$

∎

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 8n}{n^4 + 2n + 1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} 2^n \left(\frac{n}{n+1}\right)^{n^2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \sin \frac{n\pi}{4}$.

Solution. (a). Let $a_n = \frac{n^3-8n}{n^4+2n+1}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3 - 8n) \cdot n}{n^4 + 2n + 1} = \lim_{n\to\infty} \frac{1 - 8/n^2}{1 + 2/n^3 + 1/n^4} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} 1/n$ diverges by the harmonic series, the series diverges by the limit comparison test.

(b). Let $a_n = 2^n \left(\frac{n}{n+1}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} 2 \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{2}{\left(1 + \frac{1}{n}\right)^n} = \frac{2}{e} < 1.$$

The series converges by the (simplified) root test.

(c) Let $a_n = \frac{(n!)^2}{(2n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2 (2n)!}{(2n+2)!(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(1 + 1/n)^2}{(2 + 2/n)(2 + 1/n)} = \frac{1}{4} < 1.$$

The series converges by ration test.

(d) Let $a_n = \sin \frac{n\pi}{4}$. Then $\lim_{k\to\infty} a_{4k} = 0$ and $\lim_{k\to\infty} a_{8k+2} = 0$. Thus $\{a_n\}$ diverges and so the series diverges by divergence test.

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Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} [(-1)^n + 3]^n \, x^n.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{(n+1)!}(2x-1)^n.$

Solution. (a).

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|(-1)^n + 3|^n}} = \frac{1}{\overline{\lim\limits_{n\to\infty}}(-1)^n + 3} = \frac{1}{4}$$

because $\{(-1)^n + 3\} = \{2, 4, 2, 4, 2, 4, \ldots\}$ and $\overline{\lim\limits_{n\to\infty}}(-1)^n + 3 = 4.$

(b). $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{(n+1)!}(2x-1)^n = \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{(n+1)!} \cdot 2^n(x - 1/2)^n.$

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1 \cdot 3 \cdots (2n-1) \cdot (2n+1) \cdot 2^{n+1} \cdot (n+1)!}{(n+2)! \cdot 1 \cdot 3 \cdots (2n-1) \cdot 2^n}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \frac{(2n+1) \cdot 2}{(n+2)}} = \frac{1}{4}.$$

∎

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converges uniformly on the indicated intervals. Justify your answer.

$$F_n(x) = \frac{x}{1 + n^2x^2}, \quad x \in [0, 1].$$

(b) Determine whether the following series of functions converges uniformly on the indicated intervals. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2 + x}, \quad x \in [0, \infty).$$

(c) Determine the absolute convergence, conditional or divergence of the following series. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{(-1)^n \ln n}{n}.$$

Solution. (a).

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x = 0 \\ \lim\limits_{n\to\infty} \dfrac{x/n^2}{1/n^2 + x^2} = \dfrac{0}{0 + x^2} = 0 & x \neq 0 \end{cases}$$

$$T_n = \sup_{x\in[0,1]} |F_n(x) - F(x)| = \sup_{x\in[0,1]} \frac{x}{1 + n^2x^2} = \sup_{x\in[0,1]} F_n(x).$$

$$F_n'(x) = \frac{(1 + n^2x^2) - x \cdot 2n^2x}{(1 + n^2x^2)^2} = \frac{1 - n^2x^2}{(1 + n^2x^2)^2} = \begin{cases} \geq 0 & 0 \leq x \leq \frac{1}{n} \\ \leq 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

Thus

$$T_n = \sup_{x\in[0,1]} F_n(x) = F_n(1/n) = \frac{1/n}{1 + n^2 \cdot \frac{1}{n^2}} = \frac{1}{2n}.$$

Since $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, the sequence of functions converges uniformly on $[0, 1]$.

(b).

$$\left| \frac{\sin(nx)}{n^2 + x} \right| \leq \frac{1}{n^2}$$

for $x \in [0, \infty)$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The series of functions converges uniformly by Weierstrass M-test.

(c). Conditional convergence. The series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges because $\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Let $b_n = \frac{\ln n}{n}$. Then $\lim\limits_{n\to\infty} b_n = 0$. Let $f(x) = \frac{\ln x}{x}$. Then

$$f'(x) = \frac{1 - \ln x}{x^2} \leq 0 \quad \text{for} \quad x \geq e.$$

Thus $\{b_n\}$ is eventually monotone decreasing. Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by the alternating series test.

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SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Find the **interval of convergence** of the power series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) \cdot x^n.$$

Justify your answer.

(b) Let $\{a_n\}$ be a bounded sequence and let

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x - \frac{1}{2}\right)^n.$$

Prove that $f(x)$ is continuous on $[0,1]$ and

$$\int_0^1 f(x)\ dx = \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)2^{2k}}.$$

(c) Let $\{x_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} x_n$ converges. Show that $\lim_{n\to\infty} nx_n = 0$.

Solution. (a). Let $a_n = \ln\left(\frac{n+1}{n}\right)$.

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+1}\right)}{\ln\left(\frac{n+1}{n}\right)}} = \frac{1}{\lim\limits_{n\to\infty} \frac{\ln(n+2)-\ln(n+1)}{\ln(n+1)-\ln n}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \frac{\frac{1}{n+2}-\frac{1}{n+1}}{\frac{1}{n+1}-\frac{1}{n}}} = \lim_{n\to\infty} \frac{(n+2)(n+1)}{(n+1)n} = \lim_{n\to\infty} \frac{(1+2/n)(1+1/n)}{(1+1/n)} = 1.$$

Consider the ending points. When $x = 1$, the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges by limit comparison test with $\sum_{n=1}^{\infty} 1/n$ because

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)} \cdot \left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = 1.$$

When $x = -1$, the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) \cdot (-1)^n$ converges by alternating series test because $\lim_{n\to\infty} a_n = 0$ and $a_n = \ln\left(1+\frac{1}{n}\right)$ is monotone decreasing as

$$\left[\ln\left(1+\frac{1}{x}\right)\right]' = \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) \leq 0 \quad \text{for} \quad x > 0$$

(b). Since $\{a_n\}$ is bounded, there exists $M > 0$ such that $|a_n| \leq M$ for all n. Then

$$\left|a_n\left(x-\frac{1}{2}\right)^n\right| \leq M\left(\frac{1}{2}\right)^n$$

for $0 \leq x \leq 1$. Since $\sum_{n=1}^{\infty} M\left(\frac{1}{2}\right)^n$ converges, the series of functions $\sum_{n=1}^{\infty} a_n\left(x-\frac{1}{2}\right)^n$ converges uniformly on $[0,1]$ by Weierstrass M-test and so it is continuous on $[0,1]$ because each term is continuous. Moreover

$$\int_0^1 f(x)\,dx = \int_0^1 \sum_{n=1}^{\infty} a_n\left(x-\frac{1}{2}\right)^n\,dx = \sum_{n=1}^{\infty}\int_0^1 a_n\left(x-\frac{1}{2}\right)^n\,dx$$

$$= \sum_{n=1}^{\infty} a_n \left.\frac{\left(x-\frac{1}{2}\right)^{n+1}}{n+1}\right|_0^1 = \sum_{k=0}^{\infty}\frac{a_{2k}}{(2k+1)2^{2k}}.$$

Note For (b), another method can be given by showing that the power series has radius $R \geq 1$.

(c). Since $\sum_{n=1}^{\infty} x_n$ converges and $x_n \geq 0$, by Cauchy Criterion, for any $\epsilon > 0$, there exists N such that

$$x_{n+1} + \cdots + x_m = |x_{n+1} + \cdot + x_m| < \epsilon$$

for all $m > n > N$. By letting $m = 2n$,

$$\lim_{n\to\infty}(x_{n+1} + \cdots + x_{2n}) = 0.$$

From

$$0 \leq n \cdot x_{2n} \leq x_{n+1} + \cdots + x_{2n},$$

$\lim_{n\to\infty} n \cdot x_{2n} = 0$ by Squeeze Theorem and so $\lim_{n\to\infty} 2n \cdot x_{2n} = 0$. Since $\sum_{n=1}^{\infty} x_n$ converges, $\lim_{n\to\infty} x_n = 0$. From $0 \leq (2n+1)x_{2n+1} \leq (2n+1)x_{2n}$,

$\lim_{n\to\infty}(2n+1)x_{2n+1} = 0$ by Squeeze Theorem. Let $\{n_k x_{n_k}\}$ be any convergent subsequence of $\{x_n\}$. Then $\{n_k x_{n_k}\}$ converges to 0 because either $\{n_k x_{n_k}\}$ and $\{(2n)x_{2n}\}$ has a common subsequence, or $\{n_k x_{n_k}\}$ and $\{(2n+1)x_{2n+1}\}$ has a common subsequence. Thus 0 is the only subsequential limit of $\{nx_n\}$ and so $\lim_{n\to\infty} nx_n = 0$.

Note for (c), after showing that $\lim_{n\to\infty}(2n)x_{2n} = 0$, one can also use $\epsilon - N$-definition to directly conclude that $\lim_{n\to\infty} nx_n = 0$.

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Question 6 [20 marks]

(a) Using any applicable method, find the Taylor series of the function $f(x) = (x^2-1)e^{x^2}$ at $x_0 = 0$, and determine $f^{(12)}(0)$.

(b) Let $\{f_n(x)\}$ be a sequence of bounded functions on an interval I such that $\{f_n(x)\}$ converges uniformly to $f(x)$ on I. Define

$$g_n(x) = \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n}.$$

Does the sequence $\{g_n(x)\}$ converge uniformly on I? Justify your answer.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Suppose that $\lim_{n\to\infty} a_n$ exists. Show that

$$\overline{\lim_{n\to\infty}}(a_n + b_n) = \lim_{n\to\infty} a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

Solution. (a).

$$f(x) = (x^2-1)\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n+2}}{n!} - \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = -1 + \sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)x^{2n}$$

$$= -1 + \sum_{n=1}^{\infty}\frac{n-1}{n!}x^{2n}.$$

$$\frac{f^{(12)}(0)}{(12)!} = \frac{5}{6!} \implies f^{(12)}(0) = \frac{5\cdot(12)!}{6!}.$$

(b). Yes, $\{g_n\}$ converges uniformly to $f(x)$. Since f_n is bounded, there exists $M_n > 0$ such that $|f_n(x)| \le M_n$ for all $x \in I$. Since $\{f_n\}$ converges uniformly to $f(x)$, there exists n_0 such that $|f_n(x) - f_m(x)| < 1$ and $|f_n(x) - f(x)| < 1$ for $m, n > n_0$. Let $M = \max\{M_1, \ldots, M_{n_0}, M_{n_0+1} + 1\}$. Then

$$|f_n(x)| \le M$$

for all $n \geq 1$ and $x \in I$.

Given any $\epsilon > 0$, there exists N_1 such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for $n > N_1$. Let N be the smallest positive integer such that $N > \max\{\frac{2N_1M}{\epsilon}, N_1\}$. For $n > N$ and $x \in I$,

$$|g_n - f| = \left|\frac{(f_1 - f) + (f_2 - f) + \cdots + (f_n - f)}{n}\right|$$

$$\leq \frac{|f_1| + |f| + |f_2| + |f| + \cdots + |f_{N_1}| + |f|}{n} + \frac{n - N_1}{n} \cdot \frac{\epsilon}{2} < \frac{2N_1M}{n} + \frac{\epsilon}{2} < \epsilon$$

and so $\{g_n\}$ converges uniformly to $f(x)$ on I.

(c). Let $\lim_{n\to\infty} a_n = A$. There exists a subsequence $\{b_{n_k}\}$ such that $\lim_{k\to\infty} b_{n_k} = \overline{\lim_{n\to\infty}}\, b_n$. Since $\{a_n\}$ is convergent, $\lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = A$ and so

$$\lim_{k\to\infty} a_{n_k} + b_{n_k} = A + \overline{\lim_{n\to\infty}}\, b_n$$

Thus

$$\overline{\lim_{n\to\infty}}\, a_n + b_n \geq A + \overline{\lim_{n\to\infty}}\, b_n.$$

There exists a subsequence $\{a_{m_k} + b_{m_k}\}$ such that $\lim_{k\to\infty} a_{m_k} + b_{m_k} = \overline{\lim_{n\to\infty}}\, a_n + b_n$. Thus

$$\overline{\lim_{n\to\infty}}\, b_n \geq \lim_{k\to\infty} b_{m_k} = \lim_{k\to\infty} (a_{m_k} + b_{m_k}) - a_{m_k}$$

$$= \overline{\lim_{n\to\infty}}\, (a_n + b_n) - \lim_{k\to\infty} a_{m_k} = \overline{\lim_{n\to\infty}}\, (a_n + b_n) - A$$

because $\lim_{k\to\infty} a_{m_k} = \lim_{n\to\infty} a_n = A$. It follows that

$$\overline{\lim_{n\to\infty}}\, (a_n + b_n) = \overline{\lim_{n\to\infty}}\, b_n + A = \lim_{n\to\infty} a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

For (c), there are other solutions such as using definition. ∎

Question 7 [20 marks]

(a) Prove that

$$\int_0^\pi \left(\sum_{n=1}^\infty \frac{n\sin(nx)}{e^n} \right) dx = \frac{2e}{e^2-1}.$$

(b) Let $\{f_n(x)\}$ be a sequence of functions converging uniformly to 0 on I with $f_{n+1}(x) \le f_n(x)$ for $n \ge 1$ and $x \in I$. Show that the series of functions $\sum_{n=1}^\infty (-1)^n f_n(x)$ converges uniformly on I.

(c) Suppose that $a_k \ge 0$ and $\sum_{k=1}^\infty \frac{a_k}{k}$ converges. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^\infty \frac{a_k}{n+k} = 0.$$

Solution. (a). Since

$$\left| \frac{n\sin(nx)}{e^n} \right| \le \frac{n}{e^n}$$

and $\sum_{n=1}^\infty \frac{n}{e^n}$ converges by the (simplified) root test as $\lim_{n\to\infty} \sqrt[n]{\frac{n}{e^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{e} = \frac{1}{e} < 1$, the series of functions $\sum_{n=1}^\infty \frac{n\sin(nx)}{e^n}$ converges uniformly by Weierstrass M-test. Thus

$$\int_0^\pi \sum_{n=1}^\infty \frac{n\sin(nx)}{e^n}\, dx = \sum_{n=1}^\infty \int_0^\pi \frac{n\sin(nx)}{e^n}\, dx = \sum_{n=1}^\infty \frac{1-\cos(n\pi)}{e^n}$$

$$= \sum_{k=0}^\infty \frac{2}{e^{2k+1}} = \sum_{k=0}^\infty \frac{2/e}{(e^2)^k} = \frac{2/e}{1-1/e^2} = \frac{2e}{e^2-1}.$$

(b). For each $x \in I$, since $\lim_{n\to\infty} f_n(x) = 0$ (because $\{f_n\}$ pointwise converges to 0 as it uniformly converges to 0) and $f_{n+1}(x) \le f_n(x)$, we have $f_n(x) \ge 0$ and the series $\sum_{n=1}^\infty (-1)^n f_n(x)$ converges by the alternating series test. By the alternating series estimation,

$$T_n = \sup_{x\in I}\left|\sum_{k=n+1}^{\infty}(-1)^k f_k(x)\right| \le \sup_{x\in I} f_{n+1}(x).$$

Since $\{f_n(x)\}$ converges uniformly to 0 on I,

$$\lim_{n\to\infty}\sup_{x\in I} f_n(x) = \lim_{n\to\infty}\sup_{x\in I}|f_n(x)-0| = 0$$

and so $\lim_{n\to\infty} T_n = 0$ by Squeeze Theorem. Thus the series of functions $\sum_{n=1}^{\infty}(-1)^n f_n(x)$ converges uniformly.

(c). Given any $\epsilon > 0$, there exists N_1 such that

$$\sum_{k=N_1+1}^{\infty}\frac{a_k}{k} < \frac{\epsilon}{2}.$$

Let N be the positive integer such that $N > \dfrac{2\sum_{k=1}^{N_1} a_k}{\epsilon}$. For $n > N$,

$$\begin{aligned}
0 \le \sum_{k=1}^{\infty}\frac{a_k}{n+k} &= \sum_{k=1}^{N_1}\frac{a_k}{n+k} + \sum_{k=N_1+1}^{\infty}\frac{a_k}{n+k}\\
&\le \sum_{k=1}^{N_1}\frac{a_k}{n} + \sum_{k=N_1+1}^{\infty}\frac{a_k}{k} = \frac{\sum_{k=1}^{N_1} a_k}{n} + \sum_{k=N_1+1}^{\infty}\frac{a_k}{k}\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus $\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{a_k}{n+k} = 0$.

∎