Solutions to Take-home Exam 1

Question 1. [*2 points, 1 for each part*]

Prove the following limits by using $\epsilon - N$ definition

i) $\lim_{n\to\infty} \frac{3n+8}{2n+9} = \frac{3}{2}$.

ii) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$.

Proof. (i). Note that

$$\left|\frac{3n+8}{2n+9} - \frac{3}{2}\right| = \left|\frac{2(3n+8)-3(2n+9)}{2(2n+9)}\right| = \frac{11}{2(2n+9)} < \frac{11}{4n} < \frac{3}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{3}{N} \le \epsilon \iff N \ge \frac{3}{\epsilon}$. When $n > N$,

$$\left|\frac{3n+8}{2n+9} - \frac{3}{2}\right| < \frac{3}{n} < \frac{3}{N} \le \epsilon.$$

(ii). Note that

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{1}{N} \le \epsilon \iff N \ge \frac{1}{\epsilon}$. When $n > N$,

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| < \frac{1}{n} < \frac{1}{N} \le \epsilon.$$

□

Question 2. [*5 points, 1 for each part*]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\left(\sqrt{n^2+n} - n\right)\right\}$.

(b) $\left\{(2^n + 3^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}}\right\}$.

(d) $\left\{\left(\frac{3n}{3n-1}\right)^{2n+\sqrt{n}}\right\}$.

(e) $\left\{\frac{n^{50} \cdot 50^n \cdot \sin n}{n!}\right\}$.

Solution. (a).

$$\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)\cdot(\sqrt{n^2+n} + n)}{\sqrt{n^2+n}+n}$$
$$= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} = \frac{1}{2}.$$

(b).

$$\lim_{n\to\infty} (2^n+3^n)^{\frac{1}{n}} = \lim_{n\to\infty} 3\left[\left(\frac{2}{3}\right)^n + 1\right]^{\frac{1}{n}} = 3\cdot(0+1)^0 = 3.$$

Another solution: Since

$$3 = (3^n)^{\frac{1}{n}} \le (2^n+3^n)^{\frac{1}{n}} \le (3^n+3^n)^{\frac{1}{n}} = 2^{\frac{1}{n}}\cdot 3 \quad \text{and} \quad \lim_{n\to\infty} 2^{\frac{1}{n}}\cdot 3 = 3,$$

$$\lim_{n\to\infty} (2^n + 3^n)^{\frac{1}{n}} = 3 \qquad \text{by the Squeeze Thoerem.}$$

(c).

$$\lim_{n\to\infty} \sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}} = \lim_{n\to\infty} \sqrt[4]{\frac{1 + 2\dfrac{n^5}{n!} + \dfrac{\ln n}{n!}}{1 + \dfrac{5^n}{n!} + 3\dfrac{n}{n!}}} = \sqrt[4]{\frac{1+0+0}{1+0+0}} = 1.$$

(d).

$$\lim_{n\to\infty} \left(\frac{3n}{3n-1}\right)^{2n+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\left(\dfrac{3n-1}{3n}\right)^{2n+\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\left[\left(1 + \dfrac{-1}{3n}\right)^{3n}\right]^{\dfrac{2n+\sqrt{n}}{3n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\left[\left(1 + \dfrac{-1}{3n}\right)^{3n}\right]^{\dfrac{2+\frac{1}{\sqrt{n}}}{3}}} = \frac{1}{(e^{-1})^{\frac{2}{3}}} = e^{\frac{2}{3}}.$$

(e). Note that

$$-\frac{n^{50}\cdot 50^n}{n!} \le \frac{n^{50}\cdot 50^n \cdot \sin n}{n!} \le \frac{n^{50}\cdot 50^n}{n!}.$$

Since

$$\lim_{n\to\infty} \frac{n^{50}\cdot 50^n}{n!} = \lim_{n\to\infty} \frac{n^{50}}{2^n}\cdot\frac{2^n\cdot 50^n}{n!} = \lim_{n\to\infty} \frac{n^{50}}{2^n}\frac{100^n}{n!} = 0\cdot 0 = 0,$$

$$\lim_{n\to\infty} \frac{n^{50}\cdot 50^n \cdot \sin n}{n!} = 0 \qquad \text{by the Squeeze Theorem.}$$

□

Question 3. [*3 points, 1 for each part*]

(a) If $\{a_n\}$ is convergent, show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1+a_n)$ for $n \ge 1$. Assume that $\{a_n\}$ is convergent, find its limit.

(c) Find the limit of the sequence

$$\left\{\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ \ldots\right\}.$$

Solution. (a). Let $A = \lim_{n\to\infty} a_n$. Given any $\epsilon > 0$, there exists N such that

$$|a_n - A| < \epsilon$$

for all $n > N$ (by $\epsilon - N$ definition). Write b_n for a_{n+1}, that is $b_n = a_{n+1}$. For all $n > N$,

$$|b_n - A| = |a_{n+1} - A| < \epsilon$$

because $n+1 > n > N$. From $\epsilon - N$ definition, $\lim_{n\to\infty} b_n = A = \lim_{n\to\infty} a_n$, that is, $\lim_{n\to\infty} a_{n+1} = A = \lim_{n\to\infty} a_n$.

(b). Let $A = \lim_{n\to\infty} a_n$. Then

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{1+a_n} = \frac{1}{1+\lim_{n\to\infty} a_n} = \frac{1}{1+A}.$$

Thus $A(1+A) = 1$ or $A^2 + A - 1 = 0$. It follows that

$$A = \frac{-1 \pm \sqrt{5}}{2}.$$

Next we show that $a_n > 0$ by induction. When $n = 1$, $a_1 = 1 > 0$. Suppose that $a_n > 0$. Then $a_{n+1} = 1/(1 + a_n) > 0$. The induction is finished and so $a_n > 0$ for all n. It follows that $A = \lim_{n\to\infty} a_n \geq 0$. The value $\dfrac{-1-\sqrt{5}}{2}$ is rejected because $A \geq 0$, and so

$$A = \frac{-1+\sqrt{5}}{2}.$$

(c). From the sequence, we see that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. We show that $\lim_{n\to\infty} a_n$ exists:

First we prove by induction that $0 \leq a_n \leq 2$. When $n = 1$, $0 \leq a_1 \leq 2$ holds. Suppose that $0 \leq a_n \leq 2$. Then

$$0 \leq \sqrt{2a_n} = a_{n+1} \leq \sqrt{2 \cdot 2} = 2.$$

The induction is finished and so $0 \leq a_n \leq 2$ for all n.

Next since $0 \leq a_n \leq 2$, $a_{n+1} = \sqrt{2a_n} \geq \sqrt{a_n \cdot a_n} = a_n$ for all n. Thus $\{a_n\}$ is monotone increasing. By monotone convergence theorem, $\lim_{n\to\infty} a_n$ exists.

Finally let $A = \lim_{n\to\infty} a_n$. Then

$$A = \lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n} = \sqrt{2A}$$

and so $A^2 = 2A$ or $A = 0, 2$. Since $\{a_n\}$ is monotone increasing, $a_n \geq a_1 = \sqrt{2}$ for all n. It follows that $A = \lim_{n\to\infty} a_n \geq \sqrt{2}$. Thus 0 is rejected and so $A = 2$. □