2004/2005 Semester I MA2108 Advanced Calculus II

Take-home Exam 2

Question 1[2 points, 1 for each part]

Let a_1 and b_1 be positive numbers with $a_1 > b_1$. Let $a_2 = \dfrac{a_1+b_1}{2}$ be their arithmetic mean and let $b_2 = \sqrt{a_1 b_1}$ be their geometric mean. Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n+b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}.$$

(a) Show by mathematical induction that $a_n > a_{n+1} > b_{n+1} > b_n$.

(b) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Solution. (a). First we check the case $n = 1$. Since $a_1 > b_1$,

$$a_1 > \frac{a_1+b_1}{2} = a_2 > \sqrt{a_1 b_1} = b_2 > b_1,$$

where $\dfrac{a_1+b_1}{2} > \sqrt{a_1 b_1}$ because

$$a_1 + b_1 - 2\sqrt{a_1 b_1} = \left(\sqrt{a_1} - \sqrt{b_1}\right)^2 > 0.$$

Suppose that the statement holds for n, that is,

$$a_n > a_{n+1} > b_{n+1} > b_n.$$

Since $a_{n+1} > b_{n+1}$,

$$a_{n+1} > \frac{a_{n+1}+b_{n+1}}{2} = a_{n+2} > \sqrt{a_{n+1} b_{n+1}} = b_{n+2} > b_{n+1},$$

where $\dfrac{a_{n+1}+b_{n+1}}{2} > \sqrt{a_{n+1} b_{n+1}}$ because

$$a_{n+1} + b_{n+1} - 2\sqrt{a_{n+1} b_{n+1}} = \left(\sqrt{a_{n+1}} - \sqrt{b_{n+1}}\right)^2 > 0.$$

The induction is finished and so the statement holds for all n.

(b). From (a), the sequence $\{a_n\}$ is monotone decreasing and bounded below and the sequence $\{b_n\}$ is monotone increasing and bounded above. Thus $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Let $A = \lim_{n\to\infty} a_n$ and let $B = \lim_{n\to\infty} b_n$. Then

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n+b_n}{2} = \frac{\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n}{2} = \frac{A+B}{2}$$

and so $2A = A + B$ or $A = B$, that is, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. □

Question 2. [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{\dfrac{n + (-1)^n n^2}{n^2+1}\right\}$.

(b) $\{[1.5 + (-1)^n]^n\}$.

(c) $\left\{\left(1 + \dfrac{1}{n}\right)\left(1 + \sin\dfrac{n\pi}{8}\right)^{\frac{1}{n}}\right\}$

Solution. (a).

$$\frac{n+(-1)^n n^2}{n^2+1}=\frac{\dfrac{1}{n}+(-1)^n}{1+\dfrac{1}{n^2}}=\begin{cases}=\dfrac{\dfrac{1}{2k}+1}{1+\dfrac{1}{(2k)^2}} & n=2k\\[2ex] =\dfrac{\dfrac{1}{2k+1}-1}{1+\dfrac{1}{(2k+1)^2}} & n=2k+1.\end{cases}$$

Since

$$\lim_{k\to\infty}\frac{\dfrac{1}{2k}+1}{1+\dfrac{1}{(2k)^2}}=\frac{1}{1}=1 \qquad \text{and} \qquad \lim_{k\to\infty}\frac{\dfrac{1}{2k+1}-1}{1+\dfrac{1}{(2k+1)^2}}=\frac{-1}{1}=-1,$$

the set of the subsequential limits is $\{-1,1\}$, and so

$$\overline{\lim_{n\to\infty}}\frac{n+(-1)^n n^2}{n^2+1}=1 \qquad \text{and} \qquad \underline{\lim_{n\to\infty}}\frac{n+(-1)^n n^2}{n^2+1}=-1.$$

(b).

$$[1.5+(-1)^n]^n=\begin{cases}2.5^{2k} & n=2k\\ 0.5^{2k+1} & n=2k+1.\end{cases}$$

Since $\lim\limits_{k\to\infty}2.5^{2k}=+\infty$ and $\lim\limits_{k\to\infty}0.5^{2k+1}=0$,

$$\overline{\lim_{n\to\infty}}\,[1.5+(-1)^n]^n=+\infty \qquad \text{and} \qquad \underline{\lim_{n\to\infty}}\,[1.5+(-1)^n]^n=0.$$

(c). Let $n=16k+r$, where k and r are integers with $r=0,1,\cdots,15$ and $k\geq 0$. Then

$$1+\sin\frac{n\pi}{8}=1+\sin(2k\pi+\frac{r\pi}{8})=1+\sin\frac{r\pi}{8}.$$

If $r\neq 12$, then $1+\sin\dfrac{r\pi}{8}\geq 1-\dfrac{\sqrt{2}}{2}$, and so, for $r\neq 12$,

$$\begin{aligned}&\left(1+\frac{1}{16k+r}\right)\cdot\left(1-\frac{\sqrt{2}}{2}\right)^{\frac{1}{16k+r}}\\ \leq&\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}}\\ \leq&\left(1+\frac{1}{16k+r}\right)\cdot 2^{\frac{1}{16k+r}}.\end{aligned}$$

Since

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\cdot\left(1-\frac{\sqrt{2}}{2}\right)^{\frac{1}{16k+r}}=1\cdot\left(1-\frac{\sqrt{2}}{2}\right)^{0}=1$$

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\cdot 2^{\frac{1}{16k+r}}=1\cdot 2^0=1,$$

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}}=1$$

for $r\neq 12$ by the Squeeze Theorem. When $r=12$,

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\frac{1}{16k+r}}$$

$$= \lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)(1-1)^{\dfrac{1}{16k+r}} = \lim_{k\to\infty} 0 = 0.$$

Thus the set of subsequential limits is $\{0,1\}$ and so

$$\underline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 0 \quad\text{and}\quad \overline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 1.$$

Another solution of Part (c).

Since $-1 \le \sin\dfrac{n\pi}{8} \le 1$,

$$0 \le \left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} \le \left(1+\frac{1}{n}\right)2^{\frac{1}{n}}. \tag{1}$$

Thus

$$0 = \lim_{n\to\infty} 0 = \underline{\lim}_{n\to\infty} 0 \le \underline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}}.$$

On the other hand, when $n = 16k-4$ for $k = 1,2,\ldots$,

$$\sin\frac{n\pi}{8} = \sin\frac{(16k-4)\pi}{8} = \sin\left(2k\pi - \frac{\pi}{2}\right) = -1$$

and

$$\begin{aligned}&\underline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}}\\ &\le \lim_{k\to\infty}\left(1+\frac{1}{16k-4}\right)\left(1+\sin\frac{(16k-4)\pi}{8}\right)^{\frac{1}{16k-4}} = \lim_{k\to\infty} 0 = 0.\end{aligned}$$

It follows that

$$\underline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 0.$$

From Equation (1),

$$\overline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} \le \overline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)2^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)2^{\frac{1}{n}} = 1.$$

On the other hand, by choosing $n = 8k$,

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}}\\ &\ge \lim_{k\to\infty}\left(1+\frac{1}{8k}\right)\left(1+\sin\frac{8k\pi}{8}\right)^{\frac{1}{8k}} = \lim_{k\to\infty}\left(1+\frac{1}{8k}\right) = 1.\end{aligned}$$

Thus

$$\overline{\lim}_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 1.$$

□

Question 3 [5 points, 1 for each part]
Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{k}}{k^2+2k-1}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(2+\ln n)}$.

(c) $\displaystyle\sum_{n=1}^{\infty}6^n\left(1-\frac{2}{n+1}\right)^{n^2}$.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^n\cdot n!}$.

(e) $$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{k}.$$

Solution. (a). Since

$$\lim_{k\to\infty} \frac{\dfrac{\sqrt{k}}{k^2+2k-1}}{\dfrac{1}{k^{\frac{3}{2}}}} = \lim_{k\to\infty} \frac{1}{1+2/k-1/k^2} = 1,$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges by the p-series, the series converges by the limit comparison test.

(b). Let $f(x) = \dfrac{1}{x(2+\ln x)}$. Then $f(x)$ is positive and monotone decreasing on $[1, +\infty]$. Since

$$\begin{aligned}\int_1^{\infty} f(x)dx &= \int_1^{\infty} \frac{1}{x(2+\ln x)}dx \\ &= \int_2^{\infty} \frac{dy}{y} = \ln y\Big|_2^{\infty} = +\infty\end{aligned}$$

diverges, the series $\sum_{n=1}^{\infty} \frac{1}{n(2+\ln n)}$ diverges by the integral test.

(c). Let $a_n = 6^n \left(1 - \dfrac{2}{n+1}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} 6\left(1-\frac{2}{n+1}\right)^n = \lim_{n\to\infty} 6\frac{\left(1-\frac{2}{n+1}\right)^{n+1}}{1-\frac{2}{n+1}} = 6e^{-2} = \frac{6}{e^2} < 1$$

and so the series is convergent by the root test.

(d). Let $a_n = \dfrac{n^n}{3^n \cdot n!}$. Since

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}\cdot 3^n \cdot n!}{3^{n+1}\cdot(n+1)!\cdot n^n} \\ &= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n \cdot (n+1)}{3\cdot(n+1)} = \frac{e}{3} < 1,\end{aligned}$$

the series is convergent by the ratio test.

(e). Let

$$a_k = \frac{\sqrt{k+1}-\sqrt{k}}{k} = \frac{(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{k(\sqrt{k+1}+\sqrt{k})} = \frac{1}{k(\sqrt{k+1}+\sqrt{k})}$$

and let $b_k = \dfrac{1}{k^{\frac{3}{2}}}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k(\sqrt{k+1}+\sqrt{k})}}{\dfrac{1}{k^{\frac{3}{2}}}} = \lim_{k\to\infty} \frac{k^{\frac{3}{2}}}{k(\sqrt{k+1}+\sqrt{k})} = \lim_{k\to\infty} \frac{1}{\sqrt{1+1/k}+1} = \frac{1}{2}.$$

Since $\displaystyle\sum_{k=1}^{\infty} b_k$ converges by the p-series, the series converges by the limit comparison test. □