Solutions to Take-home Exam 3

Question 1. [2 points]

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, prove that $\sum_{n=1}^{\infty}(a_n+b_n)$ diverges.

(b) If a and b are positive real numbers, prove that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)^p}$$

converges if $p>1$ and diverges if $p\leq 1$.

Proof. (a). Suppose that $\sum_{n=1}^{\infty}(a_n+b_n)$ converges. Since $\sum_{n=1}^{\infty} a_n$ converges by the assumption,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}[(a_n+b_n)-a_n]$$

converges, which contradicts to the assumption that $\sum_{n=1}^{\infty} b_n$ diverges. Hence $\sum_{n=1}^{\infty}(a_n+b_n)$ converges.

(b). Let $a_k = \dfrac{1}{(ak+b)^p}$ and let $b_k = \dfrac{1}{k^p}$. Then

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{1}{(ak+b)^p}}{\frac{1}{k^p}} = \lim_{k\to\infty}\frac{k^p}{(ak+b)^p} = \lim_{k\to\infty}\frac{1}{(a+b/k)^p} = \frac{1}{a} \neq 0, \infty.$$

By the p-series, $\sum_{k=1}^{\infty} b_k$ converges if $p>1$ and diverges if $p\leq 1$. Thus, by the limit comparison test, a_k converges if $p>1$ and diverges if $p\leq 1$. □

Question 2. [3 points, 1 for each part] Test the series for convergence or divergence.

(a) $\sum_{k=1}^{\infty}(-1)^k 2^{1/k}$.

(b) $\sum_{k=1}^{\infty}\dfrac{(-1)^k k}{(k+1)(k+2)}$.

(c) $\sum_{k=1}^{\infty}(\sqrt[k]{2}-1)$.

Solution. (a). Divergence. Since $\lim_{k\to\infty} 2^{1/k}=1$, the limit $\lim_{k\to\infty}(-1)^k 2^{1/k}$ does not exist and so, by divergence test, the series diverges.

(b). Convergence. Let $b_k = \dfrac{k}{(k+1)(k+2)}$. Then $\{b_k\}$ is monotone decreasing because

$$b_k \geq b_{k+1} \iff \frac{k}{(k+1)(k+2)} \geq \frac{(k+1)}{(k+2)(k+3)}$$

$$\iff k(k+3) \geq (k+1)^2$$

$$\iff k^2+3k \geq k^2+2k+1 \iff k\geq 1.$$

Since $\lim_{k\to\infty} b_k = \lim_{k\to\infty}\dfrac{1}{(1+1/k)(k+2)} = 0$, by the alternating series test, the series converges.

(c). Let $a_k = sqrt[k]2 - 1$ and let $b_k = \dfrac{1}{k}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2^{1/k}-1}{1/k} \xlongequal{x=\frac{1}{k}} \lim_{x\to 0} \frac{2^x - 1}{x} = \lim_{x\to 0} \frac{2^x \ln 2}{1} = \ln 2.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=1}^{\infty} (\sqrt[k]{2} - 1)$ diverges. □

Question 3 [5 points, 1 for each part]
Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}+1}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2 + (-1)^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kt}{k^2+3}, \qquad t \in \mathbb{R}$.

(d) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k}+1}$.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{(k+1)^k}$.

Solution. (a). Conditional convergence. Let $a_k = \dfrac{1}{\sqrt{k}+1}$. Then $\{a_k\}$ is positive, monotone decreasing and $\lim_{k\to\infty} a_k = 0$. By the alternating series test, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}+1}$ converges. Since $\frac{1}{\sqrt{k}+1} \geq \frac{1}{\sqrt{k}+\sqrt{k}} = \frac{1}{2\sqrt{k}}$ and $\sum_{k=1}^{\infty} \frac{1}{2\sqrt{k}} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by the p-series, the series $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k}+1}\right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ diverges.

(b). Absolute convergence. Since

$$\left|\frac{(-1)^k}{2k^2+(-1)^k}\right| = \frac{1}{2k^2+(-1)^k} \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-series, the series $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{2k^2+(-1)^k}\right|$ converges by the comparison test.

(c). Absolute convergence. Since

$$\left|\frac{\sin kt}{k^2+3}\right| \leq \frac{1}{k^2+3} \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-series, the series $\sum_{k=1}^{\infty} \left|\frac{\sin kt}{k^2+3}\right|$ converges by the comparison test.

(d). Conditional convergence. Let $a_k = \dfrac{\ln(\ln k)}{\sqrt{\ln k}+1}$. Then $a_k \geq 0$ for all k. To see $\{a_k\}$ *eventually* monotone decreasing, let $f(x) = \dfrac{\ln(\ln x)}{\sqrt{\ln x}+1}$. Then

$$f'(x) = \frac{\frac{1}{\ln x}\cdot\frac{1}{x}(\sqrt{\ln x}+1) - \ln(\ln x)\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\ln x}}\cdot\frac{1}{x}}{(\sqrt{\ln x}+1)^2}$$

$$= \frac{2(\sqrt{\ln x}+1) - \ln(\ln x)\cdot\sqrt{\ln x}}{2x\ln x(\sqrt{\ln x}+1)^2} = \frac{2\left(1+\dfrac{1}{\sqrt{\ln x}}\right) - \ln(\ln x)}{2x(\ln x)^{\frac{3}{2}}(\sqrt{\ln x}+1)^2}$$

is **eventually** negative because the denominator is positive when $x > 1$, and the numerator is negative when x is large, say when $x > e^{e^4}$,

$$2\left(1+\frac{1}{\sqrt{\ln x}}\right) - \ln(\ln x) < 2(1+1) - \ln(\ln x) = 4 - \ln(\ln x) < 0.$$

Thus $\{a_k\}$ is **eventually** monotone decreasing. Now

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln(\ln k)}{\sqrt{\ln k}+1} \underset{x=\ln k}{=\!=\!=} \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}+1} = \lim_{x\to\infty} \frac{\frac{\ln x}{x^{\frac{1}{2}}}}{1+\frac{1}{\sqrt{x}}} = \frac{0}{1} = 0.$$

By the alternating series test, the series $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k}+1}$ converges. Since

$$\left|\frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k}+1}\right| = \frac{\ln(\ln k)}{\sqrt{\ln k}+1} \geq \frac{1}{\sqrt{\ln k}+1} \geq \frac{1}{\sqrt{\ln k}+\sqrt{\ln k}} = \frac{1}{2\sqrt{\ln k}} \geq \frac{1}{2\sqrt{k}}$$

for $\ln(\ln k) \geq 1$ (i.e, $k \geq e^e$) and the p-series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2\sqrt{k}} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges, the series $\displaystyle\sum_{k=2}^{\infty} \left|\frac{(-1)^k \ln(\ln k)}{\sqrt{\ln k}+1}\right|$ diverges.

(e). Divergence. Since

$$\lim_{k\to\infty} \frac{k^k}{(k+1)^k} = \lim_{k\to\infty} \frac{1}{\left(\dfrac{k+1}{k}\right)^k} = \lim_{k\to\infty} \frac{1}{\left(1+\dfrac{1}{k}\right)^k} = \frac{1}{e},$$

the limit $\displaystyle\lim_{k\to\infty} \frac{(-1)^k k^k}{(k+1)^k}$ does not exist because it has two subsequential limits $\pm\dfrac{1}{e}$. By the divergence test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^k}{(k+1)^k}$ diverges. □