

Solutions to Take-home Exam 4

Question 1 [4 points, 1 for each part]

For each of the following sequence of functions, determine whether it converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

- (a) $\left\{ \left(1 - \frac{x^2}{n}\right)^{nx} \right\}, x \in \mathbb{R}.$
 (b) $\{(\cos x)^{2n}\}, x \in \mathbb{R}.$
 (c) $\left\{ \frac{\sin nx}{\cos nx + nx} \right\}, x \in [1, +\infty).$
 (d) $\{f_n(x)\}, f_n(x) = \sum_{k=0}^n \frac{x^2}{(1+x^2)^k}, x \in \mathbb{R}.$

Solution. (a).

$$\lim_{n \rightarrow \infty} \left(1 - \frac{x^2}{n}\right)^{nx} = \lim_{n \rightarrow \infty} \left[\left(1 - \frac{x^2}{n}\right)^n \right]^x = \left(e^{-x^2}\right)^x = e^{-x^3}.$$

(b). When $x = k\pi$, then $\cos x = (-1)^k$ and so $\lim_{n \rightarrow \infty} (\cos x)^{2n} = \lim_{n \rightarrow \infty} 1 = 1$ for $x = k\pi$. When $x \neq k\pi$, the $|\cos x| < 1$ and so $\lim_{n \rightarrow \infty} (\cos x)^{2n} = 0$ in this case. Thus

$$\lim_{n \rightarrow \infty} (\cos x)^{2n} = \begin{cases} 0 & x \neq 0, \pm\pi, \pm2\pi, \pm3\pi, \dots \\ 1 & x = k\pi \text{ for some } k \in \mathbb{Z}. \end{cases}$$

(c).

$$\lim_{n \rightarrow \infty} \frac{\sin nx}{\cos nx + nx} = \lim_{n \rightarrow \infty} \frac{\frac{\sin nx}{n}}{\frac{\cos nx}{n} + x} = 0,$$

where $\lim_{n \rightarrow \infty} \frac{\sin nx}{n} = \lim_{n \rightarrow \infty} \frac{\cos nx}{n} = 0$, by the Squeeze Theorem, because

$$0 \leq \left| \frac{\sin nx}{n} \right|, \left| \frac{\cos nx}{n} \right| \leq \frac{1}{n}$$

and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

(d). When $x = 0$, $f_n(x) = 0$. In this case $\lim_{n \rightarrow \infty} f_n(x) = 0$. When $x \neq 0$,

$$\begin{aligned} f_n(x) &= \sum_{k=0}^n \frac{x^2}{(1+x^2)^k} \\ &= x^2 \left[1 + \frac{1}{1+x^2} + \dots + \left(\frac{1}{1+x^2} \right)^n \right] \\ &= x^2 \frac{1 - \left(\frac{1}{1+x^2} \right)^{n+1}}{1 - \frac{1}{1+x^2}} \\ \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} x^2 \frac{1 - \left(\frac{1}{1+x^2} \right)^{n+1}}{1 - \frac{1}{1+x^2}} = x^2 \frac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2 \end{aligned}$$

because the positive number $\frac{1}{1+x^2} < 1$ for $x \neq 0$. Thus

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & x = 0 \\ 1+x^2 & x \neq 0. \end{cases}$$

□

Question 2. [6 points, 1 for each part] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

- (a) $F_n(x) = \frac{x^n}{1+x^n}$, $x \in [0, \frac{1}{2}]$.
 (b) $F_n(x) = \frac{x^n}{1+x^n}$, $x \in [0, 1]$.
 (c) $F_n(x) = x + \frac{x}{n} \sin nx$, $x \in [-a, a]$, $a > 0$.
 (d) $F_n(x) = x + \frac{x}{n} \sin nx$, $x \in \mathbb{R}$.
 (e) $F_n(x) = \frac{x^n \sin nx}{1+x^n}$, $x \in [0, \frac{1}{2}]$.
 (f) $F_n(x) = nx(1-x^2)^n$, $x \in [0, 1]$.

Solution. (a). Uniform convergence.

$$\begin{aligned} F(x) &= \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \quad 0 \leq x \leq \frac{1}{2} \\ T_n &= \sup_{0 \leq x \leq \frac{1}{2}} |F_n(x) - F(x)| \\ &= \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1+x^n} \\ &\leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1} = \left(\frac{1}{2}\right)^n \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(b). NOT uniform convergence.

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{1+1} = \frac{1}{2} & x = 1. \end{cases}$$

Since each $F_n(x) = \frac{x^n}{1+x^n}$ is continuous on $[0, 1]$ and its limiting function $F(x)$ is NOT continuous on $[0, 1]$, the sequence of functions does not converge uniformly on $[0, 1]$.

For (c) and (d), $F(x) = \lim_{n \rightarrow \infty} x + \frac{x}{n} \sin nx = x$ because $\lim_{n \rightarrow \infty} \frac{\sin nx}{n} = 0$.

(c). Uniform convergence.

$$\begin{aligned} T_n &= \sup_{x \in [-a, a]} |F_n(x) - F(x)| \\ &= \sup_{x \in [-a, a]} \left| x + \frac{x}{n} \sin nx - x \right| \\ &= \sup_{x \in [-a, a]} \left| \frac{x}{n} \sin nx \right| \\ &\leq \sup_{x \in [-a, a]} \frac{|x|}{n} = \frac{a}{n}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{a}{n} = 0$, by the Squeeze Theorem, $\lim_{n \rightarrow \infty} T_n = 0$ and so the sequence of functions converges uniformly.

(d). NOT uniform convergence.

$$T_n = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = \sup_{x \in \mathbb{R}} \left| x + \frac{x}{n} \sin nx - x \right| = \sup_{x \in \mathbb{R}} \frac{|x| \cdot |\sin nx|}{n}$$

By choosing $x = 2n\pi + \frac{\pi}{2n}$, we have

$$\begin{aligned} T_n &\geq \frac{\left| 2n\pi + \frac{\pi}{2n} \right| \cdot \left| \sin n \left(2n\pi + \frac{\pi}{2n} \right) \right|}{n} \\ &= \frac{\left(2n\pi + \frac{\pi}{2n} \right) \cdot \left| \sin \left(2n^2\pi + \frac{\pi}{2} \right) \right|}{n} \\ &= \frac{\left(2n\pi + \frac{\pi}{2n} \right) \cdot 1}{n} \\ &= 2\pi + \frac{\pi}{2n^2} \geq 2\pi. \end{aligned}$$

Thus T_n does not tend to zero as n tends to ∞ , and so the sequence of functions does not converge uniformly.

(e). Uniform convergence. Since

$$\left| \frac{x^n \sin nx}{1 + x^n} \right| \leq \left| \frac{x^n}{1 + x^n} \right|$$

and $\lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n} = 0$ for $0 \leq x \leq \frac{1}{2}$, we have

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \frac{x^n \sin nx}{1 + x^n} = 0$$

by the Squeeze Theorem. Note that

$$\begin{aligned} T_n &= \sup_{0 \leq x \leq \frac{1}{2}} |F_n(x) - F(x)| = \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n |\sin nx|}{1 + x^n} \\ &\leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1 + x^n} \\ &\leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1} \\ &= \left(\frac{1}{2} \right)^n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(f). NOT uniform convergence. When $x = 0, 1$, $F_n(x) = 0$ and so $F(x) = \lim_{n \rightarrow \infty} F_n(x) = 0$ for $x = 0, 1$. When $0 < x < 1$, then $0 < 1 - x^2 < 1$ or $\frac{1}{1 - x^2} > 1$. Thus, for $0 < x < 1$,

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} nx (1 - x^2)^n = \lim_{n \rightarrow \infty} \frac{n}{\left(\frac{1}{1 - x^2} \right)^n} \cdot x = 0$$

by the Standard limits. Thus $F(x) = 0$ for all $0 \leq x \leq 1$. Now

$$T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} F_n(x) = \sup_{0 \leq x \leq 1} nx (1 - x^2)^n.$$

We find the maximum of $F_n(x)$ on $[0, 1]$.

$$\begin{aligned} F'_n(x) &= n(1-x^2)^n + nx \cdot n(1-x^2)^{n-1} \cdot (-2x) = 0 \\ \implies n(1-x^2)^{n-1}(1-x^2-2nx^2) &= n(1-x^2)^{n-1}(1-(2n+1)x^2) = 0 \\ \implies x &= 0, 1, \frac{1}{\sqrt{2n+1}}. \end{aligned}$$

Now

$$F_n(0) = F_n(1) = 0, \quad F_n\left(\frac{1}{\sqrt{2n+1}}\right) = n \cdot \frac{1}{\sqrt{2n+1}} \left(1 - \frac{1}{2n+1}\right)^n$$

Thus $T_n = F_n\left(\frac{1}{\sqrt{2n+1}}\right) = n \cdot \frac{1}{\sqrt{2n+1}} \left(1 - \frac{1}{2n+1}\right)^n$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} T_n &= \lim_{n \rightarrow \infty} n \cdot \frac{1}{\sqrt{2n+1}} \left(1 - \frac{1}{2n+1}\right)^n \\ &= \lim_{n \rightarrow \infty} \sqrt{n} \cdot \frac{1}{\sqrt{2+1/n}} \cdot \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1} \right]^{\frac{n}{2n+1}} \\ &= \infty \cdot \frac{1}{2} \cdot (e^{-1})^{\frac{1}{2}} = \infty. \end{aligned}$$

Hence the sequence of functions does not converge uniformly. □