2004/2005 Semester I MA2108 Advanced Calculus II

Solutions to Take-home Exam 6

Question 1. [4 points, 1 for each part] For each of the following, determine all values of x for which the given series converges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k(1-x)^k}$, $x \neq 1$.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{kx^k}$, $x \neq 0$.

Solution. (a). $\displaystyle\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k}\left(x-\frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty}\left(\frac{|-2|^k}{k}\right)^{\frac{1}{k}}} = \frac{1}{\lim\limits_{k\to\infty}\frac{2}{\sqrt[k]{k}}} = \frac{1}{2}.$$

The ending points $x = x_0 \pm R = \frac{1}{2} \pm \frac{1}{2} = 0, 1$. When $x = 0$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

diverges by the p-series. When $x = 1$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the alternating series test. In conclusion, the interval of convergence is $(0, 1]$.

(b). $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k = \sum_{k=1}^{\infty} \frac{6^k}{k^3}\left(x+\frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty}\left(\frac{6^k}{k^3}\right)^{\frac{1}{k}}} = \frac{1}{\lim\limits_{k\to\infty}\frac{6}{\left(\sqrt[k]{k}\right)^3}} = \frac{1}{6}.$$

The ending points $x = x_0 \pm R = -\frac{1}{2} \pm \frac{1}{6} = -\frac{2}{3}, -\frac{1}{3}$. When $x = -\frac{2}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^3}{k^3}$$

converges by the alternating series test. When $x = -\frac{1}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3}(2x+1)^k = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

converges by the p-series. In conclusion, the interval of convergence is $[-\frac{2}{3}, -\frac{1}{3}]$.

(c). Let $t = \frac{3x}{2(1-x)}$. Then $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k(1-x)^k} = \sum_{k=1}^{\infty} t^k$. By the geometric series, the series $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k(1-x)^k}$ converges if and only if

$$
\begin{aligned}
|t| < 1 \quad &\Longleftrightarrow \quad \left|\frac{3x}{2(1-x)}\right| < 1 \\
&\Longleftrightarrow \quad -1 < \frac{3x}{2(1-x)} < 1
\end{aligned}
$$

$$
\Longleftrightarrow \begin{cases} \frac{3x}{2(1-x)} < 1 \iff \frac{3x-2(1-x)}{2(1-x)} < 0 \iff \frac{5x-2}{2(x-1)} > 0 \iff x < \frac{2}{5} \text{ or } x > 1 \\[2ex] \frac{3x}{2(1-x)} > -1 \iff \frac{3x+2(1-x)}{2(1-x)} > 0 \iff \frac{x+2}{2(x-1)} < 0 \iff -2 < x < 1 \end{cases}
$$

$$
\Longleftrightarrow \quad -2 < x < \frac{2}{5}.
$$

Thus the answer is $-2 < x < \frac{2}{5}$.

(d). Let $t = \frac{1}{x}$. Then $\sum_{k=1}^{\infty} \frac{1}{kx^k} = \sum_{k=1}^{\infty} \frac{t^k}{k}$. The radius of convergence

$$
R = \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{\frac{1}{k}}} = 1.
$$

The ending points $t = x_0 \pm R = 0 \pm 1 = \pm 1$. When $t = 1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p-series. When $t = -1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. Thus the interval of convergence is

$$
-1 \le t < 1 \quad \Longleftrightarrow \quad -1 \le \frac{1}{x} < 1
$$

$$
\Longleftrightarrow \begin{cases} \frac{1}{x} < 1 \iff \frac{1-x}{x} < 0 \iff \frac{x-1}{x} > 0 \iff x < 0 \text{ or } x > 1 \\[2ex] \frac{1}{x} \ge -1 \iff \frac{1+x}{x} \ge 0 \iff x > 0 \text{ or } x \le -1 \end{cases}
$$

$$
\Longleftrightarrow \quad x \le -1 \text{ or } x > 1.
$$

Thus the answer is $x \le -1$ or $x > 1$. □

Question 2. [4 points, 1 for each part] Using any applicable method, find the Taylor series of each of the following functions at the indicated point, and specify the interval on which the series converges to the function.

(a) $f(x) = \cos x^2, \quad x_0 = 0.$

(b) $f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad x_0 = 0.$

(c) $f(x) = \sqrt{x}, \quad x_0 = 1.$

(d) $f(x) = \frac{x^2}{1-x^2}, \quad x_0 = 0.$

Solution. (a). Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for $x \in \mathbb{R}$,

$$
\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}
$$

and the interval is $(-\infty, \infty)$.

(b). Since

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

for $-1 < x \leq 1$, we have

$$\begin{aligned} f(x) = \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-x)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n - (-1)^{2n+1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} x^n \\ &= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \end{aligned}$$

for $-1 < x < 1$.

(c). By using the binomial series,

$$f(x) = \sqrt{x} = \sqrt{1 + (x-1)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x-1)^n$$

for $|x-1| < 1$, that is, $0 < x < 2$.

(d). By using the geometric series,

$$\begin{aligned} f(x) = \frac{x^2}{1-x^2} &= x^2 \sum_{n=0}^{\infty} \left(x^2\right)^n = x^2 \sum_{n=0}^{\infty} x^{2n} \\ &= x^2\left(1 + x^2 + x^4 + x^6 + \cdots\right) \\ &= x^2 + x^4 + x^6 + x^8 + \cdots = \sum_{n=1}^{\infty} x^{2n} \end{aligned}$$

for $|x^2| < 1$, that is, $|x| < 1$. □

Question 3. [2 points, 1 for each part]

(a) Use series to estimate the integral's value

$$\int_0^{0.1} \arctan x^2 \, dx$$

with an error of magnitude less than 10^{-8}.

(b) Let $f(x) = x^3 \sin x^9$. Find $f^{(30)}(0)$.

Solution. (a). From $\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{2n-1}$, we have

$$\arctan x^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{4n-2}}{2n-1}$$

$$\Longrightarrow \int_0^{\frac{1}{10}} \arctan x^2 \, dx = \sum_{n=1}^{\infty} \int_0^{\frac{1}{10}} \frac{(-1)^{n+1}x^{4n-2}}{2n-1} \, dx$$

$$= \sum_{n=1}^{\infty} \left. \frac{(-1)^{n+1}x^{4n-1}}{(4n-1)(2n-1)} \right|_0^{\frac{1}{10}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n-1)(2n-1)10^{4n-1}}.$$

Let $a_n = \dfrac{1}{(4n-1)(2n-1)10^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim\limits_{n\to\infty} a_n = 0$. By the alternating series estimation, from

$$a_{n+1} = \frac{1}{(4n+3)(2n+1)10^{4n+3}} < 10^{-8},$$

we have $n \geq 1$. Thus

$$\int_0^{0.1} \arctan x^2 \, dx \approx \frac{1}{3 \cdot 1 \cdot 10^3} = \frac{1}{3000}$$

with error less than 10^{-8}.

(b). Since $\sin x = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$,

$$\begin{aligned} f(x) = x^3 \sin x^9 &= x^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{9(2n-1)}}{(2n-1)!} \\ &= x^3 \left(x^9 - \frac{x^{27}}{3!} + \frac{x^{45}}{5!} - \cdots \right) \\ &= x^{12} - \frac{x^{30}}{3!} + \frac{x^{48}}{5!} - \cdots . \end{aligned}$$

By the definition of Taylor series,

$$\frac{f^{(30)}(0)}{30!} = -\frac{1}{3!}$$

and so $f^{(30)}(0) = -\dfrac{30!}{3!}$. □