

Question 1 (i).

$$R = \frac{1}{\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{|-2|^{n+1} \cdot n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}} \cdot |-2|^n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{2}{\left(1 + \frac{1}{n}\right)^{\frac{3}{2}}}} = \frac{1}{2}.$$

Consider the ending points $x = x_0 \pm R = \pm \frac{1}{2}$. The series $\sum_{n=1}^{\infty} \frac{[-2 \cdot (-\frac{1}{2})]^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent by the p -series and the series $\sum_{n=1}^{\infty} \frac{(-2 \cdot \frac{1}{2})^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ is convergent by the alternating series test. Thus the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$. \square

Question 1 (ii).

$$R = \frac{1}{\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3^{n+1} \cdot (n+1)}{(n+2) \cdot 3^n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3 \cdot \left(1 + \frac{1}{n}\right)}{1 + \frac{2}{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = 2 \pm \frac{1}{3}$. When $x = 2 - \frac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n \left(-\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

is convergent by the alternating series test. When $x = 2 + \frac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n \left(\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$$

is divergent by the p -series. Thus the interval of convergence is $\left[2 - \frac{1}{3}, 2 + \frac{1}{3}\right)$. \square

Question 1 (iii). Observe that

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n} \cdot \left(x - \frac{1}{3}\right)^n.$$

$$R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{\frac{|-3|^n}{n}}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3}{\sqrt[n]{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = \frac{1}{3} \pm \frac{1}{3}$. When $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent by the p -series. When $x = \frac{2}{3}$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent by the alternating series test. Thus the interval of convergence is $\left(0, \frac{2}{3}\right]$. \square

Question 2.

$$\begin{aligned} \sqrt[3]{9} &= (8+1)^{\frac{1}{3}} = 2 \left(1 + \frac{1}{8}\right)^{\frac{1}{3}} = 2 \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \frac{1}{8^k} \\ &= 2 + \sum_{k=1}^{\infty} \frac{\frac{1}{3} \cdot \left(\frac{1}{3} - 1\right) \cdots \left(\frac{1}{3} - k + 1\right)}{k! \cdot 2^{3k-1}} \\ &= 2 + \frac{1}{12} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3} - 1\right)}{k! \cdot 2^{3k-1}} \end{aligned}$$

Let $a_k = \frac{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3} - 1\right)}{k! \cdot 2^{3k-1}}$ for $k \geq 2$. Then

- 1) $a_k > 0$, that is, the series is alternating.
- 2) Since

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3}\right)}{(k+1)! \cdot 2^{3k+2}} \cdot \frac{k! \cdot 2^{3k-1}}{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3} - 1\right)} \\ &= \frac{k - \frac{1}{3}}{(k+1) \cdot 8} \leq 1, \end{aligned}$$

$a_{k+1} \leq a_k$ or $\{a_k\}$ is monotone decreasing.

- 3) Observe that

$$0 \leq a_k = \frac{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3} - 1\right)}{k! \cdot 2^{3k-1}} < \frac{\frac{1}{3} \cdot 1 \cdot 2 \cdots (k-1)}{k! \cdot 2^{3k-1}} = \frac{1}{3 \cdot k \cdot 2^{3k-1}}$$

for $k \geq 2$. Since $\lim_{k \rightarrow \infty} \frac{1}{3 \cdot k \cdot 2^{3k-1}} = 0$, $\lim_{k \rightarrow \infty} a_k = 0$ by the Squeeze theorem.

Thus we can apply the alternating series estimation. From

$$a_{k+1} = \frac{\frac{1}{3} \cdot \left(1 - \frac{1}{3}\right) \cdots \left(k - \frac{1}{3}\right)}{(k+1)! \cdot 2^{3k+2}} < 10^{-4},$$

we have $k \geq 3$. Thus

$$\sqrt[3]{9} \approx 2 + \frac{1}{12} - \frac{1}{288} + \frac{5}{20746} = 2.0801021214263526034$$

with an error of magnitude less than 10^{-4} . \square

Question 3.

$$\begin{aligned} \int_0^{0.1} \frac{1}{\sqrt{1+x^4}} dx &= \int_0^{\frac{1}{10}} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{\frac{1}{10}} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k} dx \\ &= \sum_{k=0}^{\infty} \int_0^{\frac{1}{10}} \binom{-\frac{1}{2}}{k} x^{4k} dx = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4k+1) \cdot 10^{4k+1}}, \end{aligned}$$

where $\int_0^{\frac{1}{10}} \sum_{k=0}^{\infty} = \sum_{k=0}^{\infty} \int_0^{\frac{1}{10}}$ because the power series $\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}$ converges at $x = 0, \frac{1}{10}$. Observe that

$$\binom{-\frac{1}{2}}{k} = (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!}.$$

The series

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4k+1) \cdot 10^{4k+1}} = \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k! \cdot (4k+1) 10^{4k+1}}$$

is an alternating series. Let $a_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k! \cdot (4k+1) 10^{4k+1}}$. Then

- (1). $a_k > 0$;
- (2). $\{a_k\}$ monotone decreasing because

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1) \cdot 2^k \cdot k! \cdot (4k+1) 10^{4k+1}}{2^{k+1} \cdot (k+1)! \cdot (4k+5) 10^{4k+5} \cdot 1 \cdot 3 \cdot 5 \cdots (2k-1)} \\ &= \frac{(2k+1) \cdot (4k+1)}{2 \cdot (k+1) \cdot (4k+5) \cdot 10^4} < 1; \end{aligned}$$

- (3). By the binomial series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges. Thus $\lim_{k \rightarrow \infty} (-1)^k a_k = 0$ and so

$$\lim_{k \rightarrow \infty} a_k = 0.$$

From

$$\frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{2^{k+1} \cdot (k+1)! \cdot (4k+5) \cdot 10^{4k+5}} < 10^{-8},$$

we have $k \geq 1$ and so

$$\int_0^{0.1} \frac{1}{\sqrt{1+x^4}} dx \approx \frac{1}{10} - \frac{1}{2 \cdot 5 \cdot 10^5} = \frac{1}{10} - \frac{1}{10^6} = 0.099999$$

with error $< 10^{-8}$. \square

Question 4. From

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x) = \sum_{k=0}^{\infty} \binom{\frac{1}{5}}{k} x^{3k},$$

by comparing the coefficients of x^{30} , we have $\frac{f^{(30)}(0)}{30!} = \binom{\frac{1}{5}}{10}$ or

$$f^{(30)}(0) = 30! \binom{\frac{1}{5}}{10} = \frac{30! \cdot \frac{1}{5} \cdot (\frac{1}{5} - 1) \cdots (\frac{1}{5} - 9)}{10!}.$$

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