2004/2005 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 2

*Question 1.* Answer: $\sup S = 1$ and $\inf S = 0$. We show that $\sup S = 1$. Let $r$ be any element in $S$. By the definition, $r$ is rational number with $0 \leq r < 1$. Thus 1 is an upper bound of $S$ and 0 is a lower bound of $S$. Let $M$ be any upper bound of $S$. Then $r \leq M$ for any rational number $r$ with $0 \leq r < 1$. In particular, $\frac{n}{n+1} \leq M$ for any positive integer $n$. It follows that $1 = \lim_{n\to\infty} \frac{n}{n+1} \leq M$ and so 1 is the least upper bound of $S$ or $\sup S = 1$. Let $m$ be any lower bound of $S$. Then $m \leq r$ for any rational number $r$ with $0 \leq r < 1$. In particular, $m \leq 0$ and so 0 is the greatest lower bound of $S$ or $\inf S = 0$. □

*Question 2.* Since $\inf B$ is a lower bound of $B$, we have $\inf B \leq x$ for any $x \in B$ and so $\inf B \leq y$ for any $y \in A \subseteq B$. It follows that $\inf B$ is a lower bound of $A$. Thus $\inf B \leq \inf A$ because $\inf A$ is the greatest lower bound of $A$. □

*Question 3 (i).* First we show that $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. Let $z$ be any element in $A \cup B$. Then $z \in A$ or $B$. If $z \in A$, then $z \leq \sup A \leq \max\{\sup A \sup B\}$. Otherwise, $z \in B$ and $z \leq \sup B \leq \max\{\sup A, \sup B\}$. Thus $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$.

Now we show that $\max\{\sup A, \sup B\}$ is the least upper bound of $A \cup B$. Let $M$ be any upper bound of $A \cup B$. Then $z \leq M$ for any $z \in A \cup B$. In particular, $z \leq M$ for $z \in A \subseteq A \cup B$ and so $M$ is an upper bound of $A$. It follows that $\sup A \leq M$. Similarly, we have $\sup B \leq M$. Thus $\max\{\sup A, \sup B\} \leq M$ and so $\max\{\sup A, \sup B\} = \sup A \cup B$. □

*Question 3 (ii).* No, it is not true. An counter-example is as follows. Let $A = \{1, 2\}$ and let $B = \{1, 3\}$. Then $\sup A = \max A = 2$ and $\sup B = \max B = 3$. It follows that $\min\{\sup A, \sup B\} = \min\{2, 3\} = 2$. But $\sup A \cap B = \sup\{1\} = 1 \neq \min\{\sup A, \sup B\}$. □

*Question 4 (i).* We prove that $2 \leq a_n \leq 3$ by induction on $n$. Since $a_1 = 2$, we have $2 \leq a_1 \leq 3$. Suppose that $a_{n-1} \leq 3$ with $n \geq 2$. Then

$$2 \leq \sqrt{6+2} \leq a_n = \sqrt{6 + a_{n-1}} \leq \sqrt{6+3} = 3.$$

The induction is finished and hence the statement.

□

*Question 4 (ii).* Let $n \geq 2$. Then

$$a_n - a_{n-1} = \sqrt{6 + a_{n-1}} - a_{n-1} = \frac{(\sqrt{6 + a_{n-1}} - a_{n-1})(\sqrt{6 + a_{n-1}} + a_{n-1})}{\sqrt{6 + a_{n-1}} + a_{n-1}}$$

$$= \frac{6 + a_{n-1} - a_{n-1}^2}{\sqrt{6 + a_{n-1}} + a_{n-1}} \geq 0$$

because $\sqrt{6 + a_{n-1}} + a_{n-1} > 0$ and $6 + x - x^2 = -(x-3)(x+2) \geq 0$ for $-2 \leq x \leq 3$. Thus $\{a_n\}$ is monotone increasing. □

*Question 4 (iii).* By (i) and (ii), $\{a_n\}$ is bounded above and monotone increasing. Thus $\{a_n\}$ is convergent. Let $A = \lim_{n\to\infty} a_n$. Then we have the equation

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{6 + a_{n-1}} = \sqrt{6 + \lim_{n\to\infty} a_{n-1}} = \sqrt{6 + A}$$

and so $A^2 = 6 + A$. It follows that $A = -2$ or $3$. Since $a_n \geq 2$ for each $n$, $A = \lim_{n\to\infty} a_n \geq 2$ and so $A = 3$.

□

*Question 5.* First we show that $0 \leq x_n \leq 1$ by induction on $n$. When $n = 1$, we have $0 \leq x_1 = \frac{3}{4} \leq 1$. Suppose that $0 \leq x_{n-1} \leq 1$ with $n \geq 2$. Observe that

$$x_n = 2x_{n-1} - x_{n-1}^2 = 1 - (1 - 2x_{n-1} + x_{n-1}^2) = 1 - (1 - x_{n-1})^2.$$

Since $0 \leq 1 - x_{n-1} \leq 1$ by induction, we have $0 \leq x_n \leq 1$. The induction is finished and so $0 \leq x_n \leq 1$ for all $n$.

Observe that

$$x_{n+1} - x_n = (2x_n - x_n^2) - x_n = x_n - x_n^2 = x_n(1 - x_n).$$

Since $0 \leq x_n \leq 1$, we have $x_{n+1} - x_n = x_n(1 - x_n) \geq 0$ and so the sequence $\{x_n\}$ is monotone increasing and bounded. Thus the limit of $\{x_n\}$ exists. Let $A = \lim_{n\to\infty} x_n$. From the equation $x_{n+1} = 2x_n - x_n^2$, we have

$$A = \lim_{n\to\infty} x_{n+1} = 2 \lim_{n\to\infty} x_n - \left(\lim_{n\to\infty} x_n\right)^2 = 2A - A^2.$$

and so $A = 0$ or $1$. Since $x_n \geq x_1 = \dfrac{3}{4}$, we have $A = \lim_{n\to\infty} x_n \geq \dfrac{3}{4}$ and so $A \neq 0$. Thus $A = 1$. $\square$

*Question 6 (a).* Since $\{a_n\} = \{4 + \cos\dfrac{n\pi}{2}\} = \{4, 3, 4, 5, 4, 3, 4, 5, 4, 3, 4, 5, \cdots\}$, we have

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 5 \quad \text{and} \quad c_n = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 3$$

for all $n$ and so $\overline{\lim_{n\to\infty}}\, 4 + \cos\dfrac{n\pi}{2} = \lim_{n\to\infty} b_n = 5$ and $\underline{\lim_{n\to\infty}}\, 4 + \cos\dfrac{n\pi}{2} = \lim_{n\to\infty} c_n = 3$. $\square$

*Question 6 (b).* Observe that

$$0 \leq \frac{1 + (-1)^n}{n} \leq \frac{2}{n}.$$

Since $\lim_{n\to\infty} \dfrac{2}{n} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} \dfrac{1 + (-1)^n}{n} = 0$ and so

$$\overline{\lim_{n\to\infty}} \frac{1 + (-1)^n}{n} = \underline{\lim_{n\to\infty}} \frac{1 + (-1)^n}{n} = \lim_{n\to\infty} \frac{1 + (-1)^n}{n} = 0.$$

$\square$