Solutions to Tutorial 3

Question 1 (a). Note that

$$a_{2k} = \frac{2-2k}{4\cdot 2k+2} \qquad \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \frac{2/k-2}{8+2/k} = -\frac{2}{8} = -\frac{1}{4}$$

$$a_{2k-1} = \frac{2+2k-1}{4\cdot(2k-1)+2} \qquad \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} \frac{1/k+2}{4\cdot(2-1/k)+2/k} = \frac{2}{8} = \frac{1}{4}$$

Thus the subsequential limits are $\pm\frac{1}{4}$ and so

$$\overline{\lim_{n\to\infty}}\, a_n = \frac{1}{4} \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n = -\frac{1}{4}.$$

□

Question 1 (b). Note that

$$a_{2k} = \left(0.9 + \sin\frac{2k\pi}{2}\right)^{2k} = 0.9^{2k} \qquad \lim_{k\to\infty} a_{2k} = 0$$

$$a_{4k+1} = \left(0.9 + \sin\frac{(4k+2)\pi}{2}\right)^{4k+2} = 1.9^{4k+2} = \left(1.9^4\right)^k \cdot 1.9^2 \qquad \lim_{k\to\infty} a_{4k+1} = +\infty$$

$$a_{4k+3} = \left(0.9 + \sin\frac{(4k+3)\pi}{2}\right)^{4k+3} = (-0.1)^{4k+3} = -0.1^{4k+3} \qquad \lim_{k\to\infty} a_{4k+3} = 0.$$

Thus the subsequential limits are 0, $+\infty$ and so

$$\overline{\lim_{n\to\infty}}\, a_n = +\infty \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n = 0.$$

□

Question 1 (c). Let $a_n = \frac{(n!)^2}{(2n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2\cdot(2n)!}{(2n+2)!\cdot(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{4}.$$

Thus

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(n!)^2}{(2n)!}} = \underline{\lim_{n\to\infty}} \sqrt[n]{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \sqrt[n]{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4}$$

□

Question 2. Let $b_n = \sup_{k \geq n} |a_k|$ and let $B_n = \sup_{k \geq n} \sqrt{|a_k|}$. Since $\sqrt{|a_k|} \leq B_n$ for $k \geq n$, we have $|a_k| \leq B_n^2$ for $k \geq n$ and so $b_n \leq B_n^2$ or $\sqrt{b_n} \leq B_n$. It follows that

$$\sqrt{\limsup_{n\to\infty} |a_n|} = \sqrt{\lim_{n\to\infty} b_n} = \lim_{n\to\infty} \sqrt{b_n} \leq \lim_{n\to\infty} B_n = \limsup_{n\to\infty} \sqrt{|a_n|}.$$

Conversely, since $|a_k| \leq b_n$ for $k \geq n$, we have $\sqrt{|a_k|} \leq \sqrt{b_n}$ for $k \geq n$ and so $B_n \leq \sqrt{b_n}$. It follows that

$$\limsup_{n\to\infty} \sqrt{|a_n|} = \lim_{n\to\infty} B_n \leq \lim_{n\to\infty} \sqrt{b_n} = \sqrt{\lim_{n\to\infty} b_n} = \sqrt{\limsup_{n\to\infty} |a_n|}$$

and hence the result.

□

Question 3. Since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Thus the limits of $\{a_n + b_n\}$ and $\{a_n b_n\}$ exist and hence $\{a_n + b_n\}$ and $\{a_n b_n\}$ are also Cauchy sequences. □

Question 4 (i). Let $a_n = \ln \frac{n+2}{n+3}$. Then the partial sum

$$\begin{aligned} S_n = a_1 + a_2 + \cdots + a_n &= \ln \frac{1+2}{1+3} + \ln \frac{2+2}{2+3} + \ln \frac{3+2}{3+3} + \cdots + \ln \frac{n+2}{n+3} \\ &= \ln \frac{(1+2)(2+2)(3+2)(4+2)\cdots(n-1+2)(n+2)}{(1+3)(2+3)(3+3)(4+3)\cdots(n-1+3)(n+3)} \\ &= \ln \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdots (n+1) \cdot (n+2)}{4 \cdot 5 \cdot 6 \cdot 7 \cdots (n+2) \cdot (n+3)} = \ln \frac{3}{n+3} = \ln 3 - \ln(n+3). \end{aligned}$$

Thus $\{S_n\}$ is divergent and so is the series $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$. □

Question 4 (ii). Let $a_n = \frac{1}{n(n+2)}$. Observe that

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

The partial sum

$$\begin{aligned} S_n &= \frac{1}{1\cdot(1+2)} + \frac{1}{2\cdot(2+2)} + \cdots + \frac{1}{n\cdot(n+2)} \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right). \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

□