

*Question 1 (a).* Let  $a_n = \frac{n^2 - 1}{2n^2 + n}$ . Then  $\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0$  and so the series  $\sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n}$  is divergent by the divergence test.  $\square$

*Question 1 (b).* Let  $a_n = \sin \frac{n\pi}{2}$ . Then  $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$  and so  $\lim_{n \rightarrow \infty} a_n$  does not exist. Thus the series  $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$  is divergent by the divergence test.  $\square$

*Question 1 (c).* Let  $a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$  and let  $b_n = \frac{1}{n}$ . Then

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + 1 + \frac{4}{n^3}}{1 + \frac{1}{n^2} + \frac{\ln n}{n^2}} = \frac{0 + 1 + 0}{1 + 0 + 0} = 1.$$

Since  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, so is  $\sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$  by the limit comparison test.  $\square$

*Question 1 (d).* Observe that

$$\frac{3 + \sin n}{n^2} \leq \frac{4}{n^2}.$$

Since  $\sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent by the  $p$ -series, the positive series  $\sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2}$  is convergent by the comparison test.  $\square$

*Question 1 (e).* Observe that

$$\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1}} = \frac{2^{n+1}}{3^n} = 2 \left(\frac{2}{3}\right)^n$$

for  $n \geq 2$ . Since  $\sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^n$  is convergent by the geometric series, the positive series

$\sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n}$  is convergent by the comparison test.  $\square$

*Question 1 (f).* Let  $a_n = \frac{2}{n^{1+\frac{1}{n}}}$  and let  $b_n = \frac{1}{n}$ . Observe that

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent by the harmonic series, the positive series  $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}}$  is divergent by the limit comparison test.  $\square$

*Question 1 (g).* Observe that

$$\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.$$

Since  $\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent by the harmonic series, the positive series  $\sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n}$  is divergent by the comparison test.  $\square$

*Question 1 (h).* Observe that

$$\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}$$

for  $p \leq 0$ . Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent by the harmonic series, the positive series  $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p}$  is divergent for  $p \leq 0$  by the comparison test.  $\square$

*Question 1 (i).* Observe that

$$\frac{n}{n^2 + 1} \geq \frac{n}{n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent by the harmonic series, the positive series  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$  is divergent by the comparison test.  $\square$

*Question 2(a).* Since  $\sum_{n=1}^{\infty} a_n$  is convergent, we have  $\lim_{n \rightarrow \infty} a_n = 0$  and so there exists a positive integer  $N$  such that  $a_n = |a_n| = |a_n - 0| < 1$  for  $n > N$ . Since  $a_n \geq 0$ ,

$$a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n$$

for  $n > N$ . By the comparison test, the positive series  $\sum_{n=1}^{\infty} a_n^2$  is convergent.  $\square$

*Another Solution of Question 2 (a).* Since  $\sum_{n=1}^{\infty} a_n$  is convergent, we have  $\lim_{n \rightarrow \infty} a_n = 0$ .

Let  $b_n = a_n^2$ . Then

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} a_n = 0,$$

that is,  $b_n \ll a_n$ . Since  $a_n \geq 0$ , by limit comparison test,  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2$  is convergent.  $\square$

*Question 2 (b).* Let  $a_n = \frac{1}{n^2}$ . Then  $\sum_{n=1}^{\infty} a_n$  is convergent but  $\sum_{n=1}^{\infty} \sqrt{a_n}$  is divergent by the  $p$ -series.  $\square$

*Question 3 (a).* Let  $f(x) = \frac{1}{x(1 + \ln x)}$ . Then  $f(x)$  is a positive monotone decreasing function over  $[1, +\infty)$ . Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x(1 + \ln x)}dx \stackrel{y=\ln x}{=} \int_0^{\infty} \frac{1}{1 + y}dy = \ln(1 + y)\Big|_0^{\infty} = +\infty.$$

is divergent, the series  $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$  is divergent by the integral test.  $\square$

*Question 3 (b).* Let  $f(x) = \frac{1}{x[1 + (\ln x)^2]}$ . Then  $f(x)$  is a positive monotone decreasing function over  $[1, +\infty)$ . Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x[1 + (\ln x)^2]}dx \stackrel{y=\ln x}{=} \int_0^{\infty} \frac{1}{1 + y^2}dy = \arctan y\Big|_0^{\infty} = \frac{\pi}{2}$$

is convergent, the series  $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$  is convergent by the integral test.  $\square$