Solutions to Tutorial 5

Question 1 (a). Let $a_n = \dfrac{(3n)!}{6^n n!(2n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[3(n+1)]!6^n n!(2n)!}{6^{n+1}(n+1)![2(n+1)]!(3n)!}$$
$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{6(n+1)(2n+2)(2n+1)} = \frac{3\cdot 3\cdot 3}{6\cdot 2\cdot 2} = \frac{27}{24} > 1.$$

Thus the series $\sum_{n=1}^{\infty} \dfrac{(3n)!}{6^n n!(2n)!}$ is divergent by the ratio test. □

Question 1 (b). Let $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} 2\left(1 - \dfrac{1}{n+1}\right)^{n+1} = \dfrac{2}{e} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent by the ratio test. □

Question 2 (a). Let $a_n = \dfrac{5n^2 \cdot 3^n}{4^{n+4}}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5^{\frac{1}{n}}(\sqrt[n]{n})^2 \cdot 3}{4\cdot 4^{\frac{4}{n}}} = \frac{1\cdot 1^2 \cdot 3}{4\cdot 1} = \frac{3}{4} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \dfrac{5n^2 \cdot 3^n}{4^{n+4}}$ is convergent by the simplified root test. □

Question 2 (b). Let $a_n = \dfrac{3^{2n}}{5^n}\left(1 - \dfrac{1}{2n}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3^2}{5}\left(1 + \frac{-\frac{1}{2}}{n}\right)^n = \frac{9}{5}e^{-\frac{1}{2}} = \frac{9}{5\sqrt{e}} > 1$$

because $e < \dfrac{9^2}{5^2} = \dfrac{81}{25} = 3.24$. Thus the series $\sum_{n=1}^{\infty} \dfrac{3^{2n}}{5^n}\left(1 - \dfrac{1}{2n}\right)^{n^2}$ is divergent by the simplified root test. □

Question 2 (c). Let a_n be the n-term in the series. Then $a_{2n-1} = \dfrac{1}{4^{2n-1}}$ and $a_{2n} = \dfrac{1}{5^{2n}}$. Thus

$$\sqrt[n]{a_n} = \begin{cases} \frac{1}{4} & \text{if} \quad \text{n is odd} \\ \frac{1}{5} & \text{if} \quad \text{n is even} \end{cases}$$

and so $\limsup_{n\to\infty} \sqrt[n]{a_n} = \dfrac{1}{4} < 1$. Hence the series is convergent by the root test. □

Question 3 (a). Let $a_n = \sqrt{2n+2} - \sqrt{n}$. Then

$$a_n = \frac{(\sqrt{2n+2} - \sqrt{n})(\sqrt{2n+2} + \sqrt{n})}{\sqrt{2n+2} + \sqrt{n}} = \frac{2n+2-n}{\sqrt{2n+2} + \sqrt{n}} = \frac{n+2}{\sqrt{2n+2} + \sqrt{n}} \to \infty$$

as $n \to \infty$. The series $\displaystyle\sum_{n=1}^{\infty} \sqrt{2n+2} - \sqrt{n}$ is divergent by the divergence test. □

Question 3 (b). Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \dfrac{2^n}{5^n}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1) \cdot 2^{n+1} \cdot n! \cdot 5^n}{(n+1)! \cdot 5^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2^n}$$

$$= \lim_{n\to\infty} \frac{(2n+1) \cdot 2}{(n+1) \cdot 5} = \frac{2 \cdot 2}{1 \cdot 5} = \frac{4}{5} < 1.$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{2^n}{5^n}$ is convergent by the ratio test. □

Question 3 (c). Let $a_n = \dfrac{\ln n}{n^{1.2}}$ and let $b_n = \dfrac{1}{n^{1.1}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n \cdot n^{1.1}}{n^{1.2}} = \lim_{n\to\infty} \frac{\ln n}{n^{0.1}} = 0.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is convergent by the p-series, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}$ is convergent by the limit comparison test for the case $a_n << b_n$. □

Question 3 (d). Let $a_n = \left(\dfrac{n}{n+2}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)^n} = \frac{1}{e^2} < 1.$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$ is convergent by the simplified root test. □

Question 3 (e). Let $a_n = \dfrac{1}{n}$ and let $b_n = \dfrac{1}{(\ln n)^3}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^3}{n} = 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$ is divergent by the limit comparison test for the case $a_n << b_n$. □

Question 3 (f). Let $a_n = \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{n}{2}}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{1}{2}} = \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3} < 1$$

and so the series is convergent by the simplified root test. □

Question 4 (i). Let $a_n = \frac{\ln n}{\sqrt{n}}$. Then $a_n \geq 0$. We show that a_n is eventually monotone decreasing. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{\frac{3}{2}}} \leq 0$$

for $x \geq e^2$ and so $\{a_n\}$ is monotone decreasing for $n \geq 9$. Since $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$ is convergent by the alternating series test. □

Question 4 (ii). Since $\left|(-1)^{n+1} \frac{\ln n}{\sqrt{n}}\right| \geq \frac{1}{n^{\frac{1}{2}}}$ for $n \geq 3$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent by the p-series, the series $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{\ln n}{\sqrt{n}}\right|$ is divergent by the comparison test. By (i), the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$ is conditionally convergent. □