NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2004-2005

MA2108/MA2108S ADVANCED CALCULUS II

November 2004 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\dfrac{1+n-3n^2}{3-2n+n^2}\right\}$.

(b) $\left\{\dfrac{\sin(n^2+1)}{n}\right\}$.

(c) $\left\{(2^n+3^n)^{1/n}\right\}$.

(d) $\left\{\left(1-\dfrac{1}{n^2}\right)^{n^2+1}\right\}$.

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3-8n}{n^4+2n+1}$.

(b) $\displaystyle\sum_{n=1}^{\infty}2^n\left(\frac{n}{n+1}\right)^{n^2}$.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$.

(d) $\displaystyle\sum_{n=1}^{\infty}\sin\frac{n\pi}{4}$.

Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $$\sum_{n=1}^{\infty} [(-1)^n + 3]^n \, x^n.$$

(b) $$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{(n+1)!} (2x-1)^n.$$

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converges uniformly on the indicated interval. Justify your answer.

$$F_n(x) = \frac{x}{1 + n^2 x^2}, \quad x \in [0, 1].$$

(b) Determine whether the following series of functions converges uniformly on the indicated interval. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2 + x}, \quad x \in [0, \infty).$$

(c) Determine the absolute convergence, conditional or divergence of the following series. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}.$$

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Find the **interval of convergence** of the power series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) \cdot x^n.$$

Justify your answer.

(b) Let $\{a_n\}$ be a bounded sequence and let

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x - \frac{1}{2}\right)^n.$$

Prove that $f(x)$ is continuous on $[0,1]$ and

$$\int_0^1 f(x)\, dx = \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)2^{2k}}.$$

(c) Let $\{x_n\}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} x_n$ converges. Show that $\lim_{n\to\infty} n x_n = 0$.

Question 6 [20 marks]

(a) Using any applicable method, find the Taylor series of the function $f(x) = (x^2-1)e^{x^2}$ at $x_0 = 0$, and determine $f^{(12)}(0)$.

(b) Let $\{f_n(x)\}$ be a sequence of bounded functions on an interval I such that $\{f_n(x)\}$ converges uniformly to $f(x)$ on I. Define

$$g_n(x) = \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n}.$$

Does the sequence $\{g_n(x)\}$ converge uniformly on I? Justify your answer.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Suppose that $\lim_{n\to\infty} a_n$ exists. Show that

$$\overline{\lim_{n\to\infty}}\,(a_n + b_n) = \lim_{n\to\infty} a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

Question 7 [20 marks]

(a) Prove that

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n} \right) \, dx = \frac{2e}{e^2 - 1}.$$

(b) Let $\{f_n(x)\}$ be a sequence of functions converging uniformly to 0 on I with $f_{n+1}(x) \leq f_n(x)$ for $n \geq 1$ and $x \in I$. Show that the series of functions $\sum_{n=1}^{\infty} (-1)^n f_n(x)$ converges uniformly on I.

(c) Suppose that $a_k \geq 0$ and $\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges. Prove that

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{a_k}{n+k} = 0.$$