INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of TWO (2) sections: Section A and Section B. It contains a total of SEVEN (7) questions and comprises FIVE (5) printed pages.

2. Answer ALL questions in Section A. Section A carries a total of 60 marks.

3. Answer no more than TWO (2) questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
SECTION A

Answer all the questions in this section. Section A carries a total of 60 marks.

Question 1 [16 marks]
For each of the following sequences, either find the limit or show that the limit does not exist.

(a) \[
\left\{ \frac{1 + n - 3n^2}{3 - 2n + n^2} \right\}.
\]
(b) \[
\left\{ \frac{\sin(n^2 + 1)}{n} \right\}.
\]
(c) \[
\left\{ (2^n + 3^n)^{1/n} \right\}.
\]
(d) \[
\left\{ \left( 1 - \frac{1}{n^2} \right)^{n^2+1} \right\}.
\]

Question 2 [16 marks]
Determine the convergence or divergence of each of the following series. Justify your answers.

(a) \[
\sum_{n=1}^{\infty} \frac{n^3 - 8n}{n^4 + 2n + 1}.
\]
(b) \[
\sum_{n=1}^{\infty} 2^n \left( \frac{n}{n + 1} \right)^n.
\]
(c) \[
\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.
\]
(d) \[
\sum_{n=1}^{\infty} \sin \frac{n\pi}{4}.
\]
**Question 3** [10 marks]
Find the radius of convergence of each of the following power series. Justify your answer.

(a) \( \sum_{n=1}^{\infty} [(-1)^n + 3]^n x^n \).

(b) \( \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{(n+1)!} (2x-1)^n \).

**Question 4** [18 marks]

(a) Determine whether the following sequence of functions converges uniformly on the indicated interval. Justify your answer.

\[ F_n(x) = \frac{x}{1 + n^2 x^2}, \quad x \in [0, 1]. \]

(b) Determine whether the following series of functions converges uniformly on the indicated interval. Justify your answer.

\[ \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2 + x}, \quad x \in [0, \infty). \]

(c) Determine the absolute convergence, conditional or divergence of the following series. Justify your answer.

\[ \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}. \]
SECTION B

Answer not more than TWO (2) questions from this section. Each question in this section carries 20 marks.

**Question 5** [20 marks]

(a) Find the interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \ln \left( \frac{n+1}{n} \right) \cdot x^n. \]
Justify your answer.

(b) Let \( \{a_n\} \) be a bounded sequence and let
\[ f(x) = \sum_{n=0}^{\infty} a_n \left( x - \frac{1}{2} \right)^n. \]
Prove that \( f(x) \) is continuous on \([0, 1]\) and
\[ \int_{0}^{1} f(x) \, dx = \sum_{k=0}^{\infty} \frac{a_{2k}}{(2k+1)2^{2k}}. \]

(c) Let \( \{x_n\} \) be a decreasing sequence of positive numbers such that \( \sum_{n=1}^{\infty} x_n \) converges. Show that \( \lim_{n \to \infty} nx_n = 0. \)

**Question 6** [20 marks]

(a) Using any applicable method, find the Taylor series of the function \( f(x) = (x^2 - 1)e^{x^2} \) at \( x_0 = 0 \), and determine \( f^{(12)}(0) \).

(b) Let \( \{f_n(x)\} \) be a sequence of bounded functions on an interval \( I \) such that \( \{f_n(x)\} \) converges uniformly to \( f(x) \) on \( I \). Define
\[ g_n(x) = \frac{f_1(x) + f_2(x) + \cdots + f_n(x)}{n}. \]
Does the sequence \( \{g_n(x)\} \) converge uniformly on \( I \)? Justify your answer.

(c) Let \( \{a_n\} \) and \( \{b_n\} \) be bounded sequences. Suppose that \( \lim_{n \to \infty} a_n \) exists. Show that
\[ \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n. \]
Question 7 [20 marks]

(a) Prove that
\[ \int_0^\pi \left( \sum_{n=1}^\infty \frac{n \sin(nx)}{e^n} \right) \, dx = \frac{2e}{e^2 - 1}. \]

(b) Let \( \{f_n(x)\} \) be a sequence of functions converging uniformly to 0 on \( I \) with \( f_{n+1}(x) \leq f_n(x) \) for \( n \geq 1 \) and \( x \in I \). Show that the series of functions \( \sum_{n=1}^\infty (-1)^n f_n(x) \) converges uniformly on \( I \).

(c) Suppose that \( a_k \geq 0 \) and \( \sum_{k=1}^\infty \frac{a_k}{k} \) converges. Prove that
\[ \lim_{n \to \infty} \sum_{k=1}^\infty \frac{a_k}{n + k} = 0. \]