

Tutorial Group:**Matriculation Number:****Name:****Signature:**

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam already.

Deadline: Monday, September 12. Hand in your papers to: the Mailbox labelled MA2108 on the 1st floor of S14. **Attach this front page to your solutions.**

Question 1 [2 points, 1 for each part]

Let a_1 and b_1 be positive numbers with $a_1 > b_1$. Let $a_2 = \frac{a_1 + b_1}{2}$ be their arithmetic mean and let $b_2 = \sqrt{a_1 b_1}$ be their geometric mean. Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n}.$$

(a) Show by mathematical induction that $a_n > a_{n+1} > b_{n+1} > b_n$.

(b) Prove that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

(**Note.** Gauss called this the common value of these limits the *arithmetic-geometric mean* of the numbers $a = a_1$ and $b = b_1$.)

Question 2. [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

- (a) $\left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\}$.
 (b) $\{ [1.5 + (-1)^n]^n \}$.
 (c) $\left\{ \left(1 + \frac{1}{n} \right) \left(1 + \sin \frac{n\pi}{8} \right)^{\frac{1}{n}} \right\}$

Question 3 [5 points, 1 for each part]

Determine the convergence or divergence of each of the following series. Justify your answers.

- (a) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1}$.
 (b) $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$.
 (c) $\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1} \right)^{n^2}$.
 (d) $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.
 (e) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k}$.