

**Tutorial Group:****Matriculation Number:****Name:****Signature:**

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam already.

**Deadline: Monday, October 24.** Hand in your papers to: the Mailbox labelled MA2108 on the 1st floor of S14. **Attach this front page to your solutions.**

**Question 1.** [6 points, 1 for each part] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a)  $\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \quad x \in [0, \infty).$

(b)  $\sum_{k=1}^{\infty} e^{-kx} x^k, \quad x \in [0, \infty).$

(c)  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}, \quad x \in [0, \infty).$

Hint: Let  $S(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+x}$  and let  $S_n(x) = \sum_{k=1}^n \frac{(-1)^{k+1}}{k+x}$ . By using alternating series estimation, show that  $T_n = \sup_{x \geq 0} |S_n(x) - S(x)| \leq \sup_{x \geq 0} \frac{1}{n+1+x} \leq \frac{1}{n+1}$ .

(d)  $\sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \quad x \in [-1, 1].$

(e)  $\sum_{n=0}^{\infty} \left( \frac{1}{nx+2} - \frac{1}{(n+1)x+2} \right), \quad x \in [0, 1].$

(Hint: Find the partial sums  $S_n(x) = \sum_{k=0}^n \left( \frac{1}{kx+2} - \frac{1}{(k+1)x+2} \right)$  and then show that  $S(x) = \lim_{n \rightarrow \infty} S_n(x)$  is not continuous while each  $S_n(x)$ . From this, conclude that the series of functions does not converge uniformly.)

(f)  $\sum_{k=1}^{\infty} \left( \frac{x}{2} \right)^k, \quad x \in (-2, 2).$

(Hint: Show that  $T_n = \sup_{-2 < x < 2} \left| \sum_{k=n+1}^{\infty} \left( \frac{x}{2} \right)^k \right| = +\infty$  by letting  $x \rightarrow 2^-$ .)

**Question 2** [4 points, 1 for each part] Find the radius of convergence of each of the following power series:

(a)  $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k.$

(b)  $\sum_{k=1}^{\infty} \left( 1 - \frac{1}{k} \right)^{k^2} (x-1)^k.$

(c)  $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k.$

(d)  $\sum_{n=0}^{\infty} n \left( \frac{x}{2} \right)^{n^2}.$

(Hint. For part (d), you can use the root test to find the interval for which the series converges absolutely.)