

1. Prove the following limits by using  $\epsilon - N$  definition

i)  $\lim_{n \rightarrow \infty} \frac{2n^2}{3n^2 + 2} = \frac{2}{3}$ ;

ii)  $\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1$ .

2. For each of the following statements, either prove that the statement is true or give a counter example to show that the statement is false:

i) If  $\{a_n\}$  converges and  $\{b_n\}$  diverges, then  $\{a_n + b_n\}$  diverges.  
(Hint: Use Theorem 1.4.5)

ii) If  $\{a_n\}$  converges and  $\{b_n\}$  diverges, then  $\{a_n b_n\}$  diverges.  
(Hint: Construct a counter-example.)

[From Question 3 onwards, you may assume the limits of the standard sequences.]

3. Evaluate the following limits:

i)  $\lim_{n \rightarrow \infty} \frac{n^2 + 5n^3 - 1}{3n^3 + 6n + 4}$ ;

ii)  $\lim_{n \rightarrow \infty} \frac{3^n + n^8}{2n^2 + 7^n}$ ;

iii)  $\lim_{n \rightarrow \infty} \sqrt{\frac{n^4 + 4n^3 + 1}{n^3 + 2n^2}}$ .

4. Use the Squeeze theorem to find the following limits:

i)  $\lim_{n \rightarrow \infty} \frac{1 + |\sin n|}{2n}$ ;

ii)  $\lim_{n \rightarrow \infty} \left( \frac{2n - 5}{3n + 1} \right)^n$ .

5. Do the following sequences tend to  $+\infty$  or  $-\infty$ ? Justify your answer.

i)  $\left\{ \frac{e^n}{n^{100}} \right\}$ ;

ii)  $\left\{ \frac{n}{\ln \frac{1}{n+2}} \right\}$ .

6. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

(a).  $\lim_{n \rightarrow \infty} \sqrt[3]{\frac{2n^4 + n + 1}{16n^4 + n^2 + 2}}$ ;

(b).  $\lim_{n \rightarrow \infty} \left( 3 + \ln\left(\cos \frac{1}{\sqrt{n}}\right) + \frac{n^2}{1.1^n} \right)$ ;

(c).  $\lim_{n \rightarrow \infty} \frac{n^4 + 8^n}{9^n + n + 8^n}$ ;

- (d).  $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3}}$ ;
- (e).  $\lim_{n \rightarrow \infty} (\sqrt{3n-2} - \sqrt{3n-3})$ ;
- (f).  $\lim_{n \rightarrow \infty} \left( \frac{3 + (-1)^n}{5} \right)^n$ ;
- (g).  $\lim_{n \rightarrow \infty} \frac{7^n + \ln n - n!}{n! + n^2}$ ;
- (h).  $\lim_{n \rightarrow \infty} \frac{n^{100} 100^n}{n!}$ ;
- (i).  $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$ ;
- (j).  $\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}}$ ;
- (k).  $\lim_{n \rightarrow \infty} n \sin \frac{3}{n}$ ; (Hint:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .)
- (l).  $\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{3n} \right)^{2n}$ ;
- (m).  $\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n^3} \right)^{n^3+2}$ .

7. Let  $S$  and  $T$  be two bounded sets of real numbers. Show that  $S \cup T$  is also a bounded set.

(Hint: Recall that a bounded set means that this set has an upper bound and a lower bound. The assumption says that  $S$  has an upper bound and a lower bound, and  $T$  has a (possibly different) upper bound and a (possibly different) lower bound. What you need to do is to find an upper bound and a lower bound for the union  $S \cup T$ , that is, a **common** upper bound and a **common** lower bound for both  $S$  and  $T$ . You also have to think how to write down your solution in a *logical* way.)

8. i) Show that a sequence  $\{a_n\}$  is bounded if and only if  $\{|a_n|\}$  is bounded.

(Hint: Try to think: Whence you have an upper bound and a lower bound for  $\{a_n\}$ , how to give an upper bound and a lower bound for  $\{|a_n|\}$ , and vice versa.)

ii) Using i) or otherwise, show that if  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\{b_n\}$  is bounded, then  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .

(Hint: From (i),  $\{|b_n|\}$  has an upper bound. Then try  $\epsilon - N$  definition.)