

1. Show that the function  $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$  is differentiable on  $(-\infty, +\infty)$ .
2. (a). Prove that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (|x| < \infty)$$

is a solution of the differential equation

$$\frac{dy}{dx} = y.$$

- (b). By solving the above differential equation, show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (|x| < \infty).$$

3. By integrating from  $t = 0$  to  $t = x$  the power series  $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$ ,  $|t| < 1$ , show that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

for all  $|x| < 1$ .

4. Find the radius of convergence of each of the following power series:

- i)  $\sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^{n^2} x^n$ .

- ii)  $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} (x-1)^n$ .

- iii)  $\frac{x}{5} + \left(\frac{x}{6}\right)^2 + \left(\frac{x}{5}\right)^3 + \left(\frac{x}{6}\right)^4 + \left(\frac{x}{5}\right)^5 + \left(\frac{x}{6}\right)^6 + \cdots$ .

- iv)  $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2}$ .