Problem-Based Learning for Advanced Calculus II

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Proposed Problem: Computation of π

- **Step (1).** Recall the formula

$$\frac{\pi}{6} = \arcsin \frac{1}{2}.$$

- **Step (2).** Recall the integral formula that

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x \left(1-t^2\right)^{-\frac{1}{2}}\, dt.$$

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Binomial Series

- Recall the formula

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

- **Generalization: Binomial Series**

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots$$

holds for any real number α,where the binomial number

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha-1) \cdot \cdots \cdot (\alpha-k+1)}{k!}.$$

- We will prove that this formula holds when $|x| < 1$.

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Notes on Binomial Series

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  $$\binom{\alpha}{k} = \frac{\alpha\cdot(\alpha-1)\cdot\cdots\cdot(\alpha-k+1)}{k!}.$$

- If α is **NOT** a non-negative integer, the **binomial number** $\binom{\alpha}{k}$ is never zero and so the right hand side of the above formula is an **infinite summation**, which is called **(infinite) series** that we are going to study in this module.

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$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x \left(1-t^2\right)^{-\frac{1}{2}}\, dt.$$

- From binomial series, by letting $\alpha = -\frac{1}{2}$ and replacing x to be $-t^2$, obtain the formula

$$(1-t^2)^{-\frac{1}{2}} = 1 + \binom{-\frac{1}{2}}{1}(-t^2) + \binom{-\frac{1}{2}}{2}(-t^2)^2 + \binom{-\frac{1}{2}}{3}(-t^2)^3 + \cdots.$$

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- **Step (3).**

$$\frac{\pi}{6} = \int_0^{\frac{1}{2}} (1 - t^2)^{-\frac{1}{2}}\ dt$$

$$= \int_0^{\frac{1}{2}} \left[1 + \binom{-\frac{1}{2}}{1}(-t^2) + \binom{-\frac{1}{2}}{2}(-t^2)^2 + \binom{-\frac{1}{2}}{3}(-t^2)^3 + \cdots \right] dt$$

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Formula

- By working out each of the above integrals, there is a (conjectured) formula

$$\frac{\pi}{6} = \frac{1}{2} - \binom{-\frac{1}{2}}{1}\frac{1}{3\cdot 2^3} + \binom{-\frac{1}{2}}{2}\frac{1}{5\cdot 2^5} - \binom{-\frac{1}{2}}{3}\frac{1}{7\cdot 2^7} + \cdots$$

$$= \frac{1}{2} + \frac{1}{3\cdot 2^4} + \frac{(-\frac{1}{2})\cdot(-\frac{3}{2})}{2\cdot 1}\frac{1}{5\cdot 2^5} - \frac{(-\frac{1}{2})\cdot(-\frac{3}{2})\cdot(-\frac{5}{2})}{3\cdot 2\cdot 1}\frac{1}{7\cdot 2^7} + \cdots$$

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Add up first few terms

- By only taking the first term, we compute

$$\pi \approx 6 \cdot \frac{1}{2} = 3.$$

- By taking the first two terms,

$$\pi \approx 6 \cdot \left(\frac{1}{2} + \frac{1}{3 \cdot 2^4} \right) = 3 + \frac{1}{8} = 3.125,$$

which is more accurate.

- By taking the first three terms,

$$\pi \approx 6 \cdot \left(\frac{1}{2} + \frac{1}{3 \cdot 2^4} + \frac{3}{5 \cdot 2^8} \right) = 3 + \frac{1}{8} + \frac{9}{640} = 3.1390625,$$

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Help from Computer

- Let's ask the computer to check this formula. By adding up first 10-terms, Maple answers that we get 8 digits correct for π.

- In the last lecture, we will fill-in the proofs for the above computation.

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What we are going to learn

Arising from this problem, we need to learn:

- How can we know an infinite series adding up to a finite number, namely**Convergence**/**Divergence**?
- How to obtain the binomial series?Can we write down other functions like $\sin x$, $\cos x$, e^x as an infinite summation of powers of x?These will be answered by studying **sequences and series of functions**, **power series**,**Taylor series** and etc.
- For an infinite summation of functions, can we do integrals and derivatives term-by-term? The answer is that **sometimes is yes and sometimes is no.** One needs a condition so-called **uniform convergence**.

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