MA2108 **Midterm** **11 October 2005**

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Monday 1-2 (ST1)	Monday 2-3 (ST2)	Monday 3-4 (ST3)
Monday 4-6 (ST4)	Wednesday 9-10 (ST5)	Wednesday 10-11 (ST6)
Wednesday 11-12 (ST7)		Wednesday 12-1 (ST8)

Name:______________________________

Matriculation number:______________________________

INSTRUCTIONS TO CANDIDATES

1. This test contains a total of **SIX (6)** Questions.
2. Answer **ALL** six questions. Show your steps.
3. The examination carries a total of 50 marks.
4. Candidates may use a help-sheet, up to A4 size (both sided).

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Problem #	Your Grades
1 (15 points)	
2 (5 points)	
3 (15 points)	
4 (5 points)	
5 (5 points)	
6 (5 points)	
total (50 points)	

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1. [15 marks] Determine the limit of the following sequences. Justify your answer.

(a) $\left\{ \dfrac{n^3 - n - 28}{3n^3 + 65n^2 + 69} \right\}$.

$$\lim_{n\to\infty} \frac{n^3 - n - 28}{3n^3 + 65n^2 + 69} = \lim_{n\to\infty} \frac{1 - 1/n^2 - 28/n^3}{3 + 65/n + 69/n^3} = \frac{1}{3}.$$

(b) $\left\{ \dfrac{1}{\sqrt{9^n + 3^{n+2}} - \sqrt{9^n - n^2}} \right\}$.

$$\lim_{n\to\infty} \frac{1}{\sqrt{9^n + 3^{n+2}} - \sqrt{9^n - n^2}} = \lim_{n\to\infty} \frac{\sqrt{9^n + 3^{n+2}} + \sqrt{9^n - n^2}}{3^{n+2} + n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1 + 3^{n+2}/9^n} + \sqrt{1 - n^2/9^n}}{3^2 + n^2/3^n} = \frac{2}{9}$$

(c) $\left\{ (n^2 + n)^{\frac{1}{1+2\ln n}} \right\}$.

Let $a_n = (n^2 + n)^{\frac{1}{1+2\ln n}}$. Then

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(n^2 + n)}{1 + 2\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+n} \cdot (2n+1)}{2 \cdot \frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2n^2 + n}{2(n^2 + n)} = \lim_{n\to\infty} \frac{2 + 1/n}{2(1 + 1/n)} = 1.$$

Thus $\lim\limits_{n\to\infty} (n^2 + n)^{\frac{1}{1+2\ln n}} = e$.

2. [5 marks] Determine the limit of the sequence $\left\{ \frac{\sqrt[n]{(3n)!}}{n^3} \right\}$. Justify your answer.

Solution. Let $a_n = \frac{(3n)!}{n^{3n}}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(3n+3)!}{(n+1)^{3n+3}} \cdot \frac{n^{3n}}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} \cdot \frac{1}{(1+1/n)^{3n}} = \frac{3^3}{e^3}.$$

Thus

$$\lim_{n\to\infty} \frac{\sqrt[n]{(3n)!}}{n^3} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3^3}{e^3}.$$

□

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3. [15 marks] Determine convergence or divergence of the following series. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{3^n + 2^{n+1}}{5^n + n^2}}$.

Let $a_n = \sqrt{\dfrac{3^n + 2^{n+1}}{5^n + n^2}}$ and let $b_n = \left(\sqrt{\dfrac{3}{5}}\right)^n$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{\frac{3^n + 2^{n+1}}{5^n + n^2}} \cdot \sqrt{\frac{5^n}{3^n}} = \lim_{n\to\infty} \sqrt{\frac{1 + 2\cdot\left(\frac{2}{3}\right)^n}{1 + n^2/5^n}} = 1$$

Since $\sum_{n=1}^{\infty} b_n$ converges, $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{3^n + 2^{n+1}}{5^n + n^2}}$ converges by limit comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n}{3n+1}$.

Let $a_n = (-1)^n \frac{3n}{3n+1}$. Since $\lim_{n\to\infty} \frac{3n}{3n+1} = 1$, $\lim_{n\to\infty} a_n$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{3n+1}$ diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{[(2n)!]^2 2^{n+2}}{n!(3n)!}$.

Let $a_n = \dfrac{[(2n)!]^2 2^{n+2}}{n!(3n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(2n+2)!]^2 2^{n+3}}{(n+1)!(3n+3)!} \cdot \frac{n!(3n)!}{[(2n)!]^2 2^{n+2}} = \lim_{n\to\infty} 2\frac{(2n+2)^2(2n+1)^2}{(n+1)(3n+3)(3n+2)(3n+1)}$$

$$= \lim_{n\to\infty} 2\frac{(2+2/n)^2(2+1/n)^2}{(1+1/n)(3+3/n)(3+2/n)(3+1/n)} = \frac{32}{27} > 1.$$

The series diverges by ratio test.

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4. [5 marks] Determine the absolute convergence, conditional convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n+2\ln n}$. Justify your answer.

Solution. Let $b_n = \frac{\ln n}{n+2\ln n}$. Then $b_n \geq 0$. Let $f(x) = \frac{\ln x}{x+2\ln x}$. Then

$$f'(x) = \frac{\frac{1}{x}\cdot(x+2\ln x) - \ln x\cdot(1+2\frac{1}{x})}{(x+2\ln x)^2} = \frac{1-\ln x}{(x+2\ln x)^2} \leq 0$$

for $x \geq e$. Thus b_n is eventually monotone decreasing. Since

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\frac{\ln n}{n}}{1+2\frac{\ln n}{n}} = 0,$$

the series converges by the alternating series test.

Let $a_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{n+2\ln n}{\ln n}\cdot\frac{1}{n} = \frac{1+2\frac{\ln n}{n}}{\ln n} = 0.$$

Since $\sum\limits_{n=1}^{\infty} a_n$ diverges by p-series, $\sum\limits_{n=1}^{\infty} b_n$ diverges by limit comparison test. Thus the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n+2\ln n}$ converges conditionally. □

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5. [5 marks] Let A be a non-empty bounded set of non-negative real numbers such that $\sup A = 1$. Consider the set

$$C = \left\{\sqrt{a+b+2} \, : \, a, b \in A\right\}.$$

Find the value of $\sup C$. Justify your answer.

solution. Given any $a, b \in A$. Since $0 \leq a, b \leq 1$,

$$\sqrt{a+b+2} \leq \sqrt{1+1+2} = 2.$$

Thus 2 is an upper bound of C. Let M be any upper bound of C. Then

$$\sqrt{a+b+2} \leq M$$

for any $a, b \in A$. In particular, for any $a \in A$,

$$\sqrt{a+a+2} \leq M \Longrightarrow 2a+2 \leq M^2 \Longrightarrow a \leq \frac{M^2-2}{2}.$$

Thus $\dfrac{M^2-2}{2}$ is an upper bound of A and so

$$1 = \sup A \leq \frac{M^2-2}{2} \Longrightarrow 2 \leq M^2 - 2 \Longrightarrow 4 \leq M^2$$
$$\Longrightarrow M \geq 2 \text{ or } M \leq -2$$

Since $M \geq 0$, $M \geq 2$. Thus $\sup C = 2$.

□

6. [5 marks] Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{a_n}$ is convergent. Is it true that the series $\sum_{n=1}^{\infty} \dfrac{a_n^3}{n^4}$ is divergent? Justify your answer.

Solution. Since $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{a_n}$ converges, the n-term $(-1)^n \frac{n}{a_n}$ tends to 0 and so

$$\lim_{n\to\infty} \frac{n}{a_n} = \lim_{n\to\infty} \left|(-1)^n \frac{n}{a_n}\right| = \left|\lim_{n\to\infty} (-1)^n \frac{n}{a_n}\right| = 0.$$

Let $b_n = \frac{a_n^3}{n^4}$ and let $c_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^4}{a_n^3} = \lim_{n\to\infty} \left(\frac{n}{a_n}\right)^3 = 0.$$

Since $\sum_{n=1}^{\infty} c_n$ diverges by p-series, the series $\sum_{n=1}^{\infty} \frac{a_n^3}{n^4}$ diverges by limit comparison test.

□