2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Take-home Exam 1

1. Use the $\epsilon - N$ definition to prove that $\lim\limits_{n\to\infty}\dfrac{3n^2+8}{2n^2+9}=\dfrac{3}{2}$.

Proof. Note that

$$\left|\frac{3n^2+8}{2n^2+9}-\frac{3}{2}\right|=\left|\frac{2(3n^2+8)-3(2n^2+9)}{2(2n^2+9)}\right|=\frac{11}{2(2n^2+9)}<\frac{11}{4n^2}<\frac{3}{n}.$$

Given $\epsilon>0$, choose N such that $\dfrac{3}{N}\le\epsilon \iff N\ge\dfrac{3}{\epsilon}$. When $n>N$,

$$\left|\frac{3n^2+8}{2n^2+9}-\frac{3}{2}\right|<\frac{3}{n}<\frac{3}{N}\le\epsilon.$$

$\square$

2. For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\left(\sqrt{n^2+2n}-n\right)\right\}$.

(b) $\left\{(5^n+3^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\sqrt[5]{\dfrac{n!-2n^5+\ln n}{n!+5^n+3n}}\right\}$.

(d) $\left\{\left(\dfrac{3n}{3n-1}\right)^{2n+\sqrt[3]{n}}\right\}$.

(e) $\left\{\dfrac{n^{50}\cdot 50^n\cdot\cos n}{n!}\right\}$.

(f) $\left\{\left(\sqrt{4-\dfrac{3}{n}}-2\right)n\right\}$.

Solution. (a).

$$\lim_{n\to\infty}\left(\sqrt{n^2+2n}-n\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+2n}-n\right)\cdot\left(\sqrt{n^2+2n}+n\right)}{\sqrt{n^2+2n}+n}$$

$$=\lim_{n\to\infty}\frac{n^2+2n-n^2}{\sqrt{n^2+2n}+n}=\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+2n}+n}=\lim_{n\to\infty}\frac{2}{\sqrt{1+2/n}+1}=1.$$

(b).

$$\lim_{n\to\infty}(5^n+3^n)^{\frac{1}{n}}=\lim_{n\to\infty}5\left[1+\left(\frac{3}{5}\right)^n\right]^{\frac{1}{n}}=5\cdot(0+1)^0=5.$$

Another solution: Since

$$5=(5^n)^{\frac{1}{n}}\le(5^n+3^n)^{\frac{1}{n}}\le(5^n+5^n)^{\frac{1}{n}}=2^{\frac{1}{n}}\cdot 5\quad\text{and}\quad\lim_{n\to\infty}2^{\frac{1}{n}}\cdot 5=5,$$

$$\lim_{n\to\infty}(5^n+3^n)^{\frac{1}{n}}=5\qquad\text{by the Squeeze Thoerem.}$$

(c).

$$\lim_{n\to\infty}\sqrt[5]{\frac{n!-2n^5+\ln n}{n!+5^n+3n}}=\lim_{n\to\infty}\sqrt[5]{\frac{1-2\dfrac{n^5}{n!}+\dfrac{\ln n}{n!}}{1+\dfrac{5^n}{n!}+3\dfrac{n}{n!}}}=\sqrt[5]{\frac{1+0+0}{1+0+0}}=1.$$

(d).

$$\lim_{n\to\infty}\left(\frac{3n}{3n-1}\right)^{2n+\sqrt[3]{n}} = \lim_{n\to\infty}\frac{1}{\left(\dfrac{3n-1}{3n}\right)^{2n+\sqrt[3]{n}}} = \lim_{n\to\infty}\frac{1}{\left[\left(1+\dfrac{-1}{3n}\right)^{3n}\right]^{\dfrac{2n+\sqrt[3]{n}}{3n}}}$$

$$= \lim_{n\to\infty}\frac{1}{\left[\left(1+\dfrac{-1}{3n}\right)^{3n}\right]^{\dfrac{2+\frac{1}{\sqrt[3]{n^2}}}{3}}} = \frac{1}{(e^{-1})^{\frac{2}{3}}} = e^{\frac{2}{3}}.$$

(e). Note that

$$-\frac{n^{50}\cdot 50^n}{n!} \le \frac{n^{50}\cdot 50^n\cdot\cos n}{n!} \le \frac{n^{50}\cdot 50^n}{n!}.$$

Since

$$\lim_{n\to\infty}\frac{n^{50}\cdot 50^n}{n!} = \lim_{n\to\infty}\frac{n^{50}}{2^n}\cdot\frac{2^n\cdot 50^n}{n!} = \lim_{n\to\infty}\frac{n^{50}}{2^n}\frac{100^n}{n!} = 0\cdot 0 = 0,$$

$$\lim_{n\to\infty}\frac{n^{50}\cdot 50^n\cdot\cos n}{n!} = 0 \qquad \text{by the Squeeze Theorem.}$$

(f).

$$\lim_{n\to\infty}\left(\sqrt{4-\frac{3}{n}}-2\right)n = \lim_{n\to\infty}\frac{\left(4-\frac{3}{n}-4\right)n}{\sqrt{4-\frac{3}{n}}+2} = \lim_{n\to\infty}\frac{-3}{\sqrt{4-\frac{3}{n}}+2} = \frac{-3}{\sqrt{4-0}+2} = -\frac{3}{4}.$$

□

3. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Prove that

(i) The sequence $\{|a_n-b_n|\}$ is convergent. [Hint: the function $f(x)=|x|$ is continuous.]

(ii) From (i) or otherwise, prove that the sequence $\{c_n\}$ defined by $c_n=\max\{a_n,b_n\}$ is also convergent. [Hint: First show that $\max\{a,b\}=\frac{1}{2}\left(a+b+|a-b|\right)$.]

Solution. (i). Since both $\{a_n\}$ and $\{b_n\}$ are convergent, the sequence $\{a_n-b_n\}$ is convergent. Let $f(x)=|x|$. Then $f(x)$ is continuous. Thus the sequence $\{|a_n-b_n|\}=\{f(a_n-b_n)\}$ is also convergent.

(ii). Note that

$$\frac{1}{2}\left(a+b+|a-b|\right) = \begin{cases} \frac{1}{2}\left(a+b+a-b\right)=a & \text{if} \quad a\ge b \\ \frac{1}{2}\left(a+b+b-a\right)=b & \text{if} \quad b\ge a \end{cases}$$

Thus $\max\{a,b\}=\frac{1}{2}\left(a+b+|a-b|\right)$ for any a,b. Hence $c_n=\frac{1}{2}\left(a_n+b_n+|a_n-b_n|\right)$ for each n. From (i), the sequence $\{|a_n-b_n|\}$ converges. Thus the sequence

$$\{c_n\}=\left\{\frac{1}{2}\left(a+b+|a-b|\right)\right\}$$

converges.

□