NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Take-home Exam 2

1. Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{\dfrac{n+(-1)^n n^2}{n^2+1}\right\}$.

(b) $\left\{(1+(-1)^n)\sin\dfrac{n\pi}{4}\right\}$

(c) $\sqrt[n]{\dfrac{(2n)!}{(n!)^2}}$. [Hint: Recall Proposition 1.8.15 in lecture notes]

Solution. (a).

$$\frac{n+(-1)^n n^2}{n^2+1} = \frac{\frac{1}{n}+(-1)^n}{1+\frac{1}{n^2}} = \begin{cases} = \dfrac{\frac{1}{2k}+1}{1+\frac{1}{(2k)^2}} & n=2k \\[2ex] = \dfrac{\frac{1}{2k+1}-1}{1+\frac{1}{(2k+1)^2}} & n=2k+1. \end{cases}$$

Since

$$\lim_{k\to\infty}\frac{\frac{1}{2k}+1}{1+\frac{1}{(2k)^2}} = \frac{1}{1} = 1 \qquad \text{and} \qquad \lim_{k\to\infty}\frac{\frac{1}{2k+1}-1}{1+\frac{1}{(2k+1)^2}} = \frac{-1}{1} = -1,$$

the set of the subsequential limits is $\{-1,1\}$, and so

$$\overline{\lim_{n\to\infty}}\frac{n+(-1)^n n^2}{n^2+1} = 1 \qquad \text{and} \qquad \underline{\lim_{n\to\infty}}\frac{n+(-1)^n n^2}{n^2+1} = -1.$$

(b).

$$(1+(-1)^n)\sin\frac{n\pi}{4} = \begin{cases} 0 & n=8k,8k+1,8k+3,8k+4,8k+5,8k+7 \\ 2\sin\left(2k\pi+\frac{\pi}{2}\right)=2 & n=8k+2 \\ 2\sin\left(2k\pi+\frac{3\pi}{2}\right)=-2 & n=8k+6 \end{cases}$$

the set of the subsequential limits is $\{-2,0,2\}$, and so

$$\overline{\lim_{n\to\infty}}\,(1+(-1)^n)\sin\frac{\pi}{4} = 2 \qquad \text{and} \qquad \underline{\lim_{n\to\infty}}\,(1+(-1)^n)\sin\frac{\pi}{4} = -2.$$

(c). Let $a_n = \dfrac{(2n)!}{(n!)^2}$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+2)!\cdot(n!)^2}{[(n+1)!]^2\cdot(2n)!} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2} = 4.$$

Thus

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{(n!)^2}} = \underline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$$

□

2. Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{a_n}{a_n + 1} \quad \text{for } n \geq 1.$$

Prove that $\{a_n\}$ is convergent, and find its limit.

Solution. First we prove that $1 \leq a_n \leq 2$ by induction. This is true for $n = 1$. Assume that $1 \leq a_n \leq 2$. Then

$$1 \leq 1 + \frac{a_n}{a_n + 1} = a_{n+1} \leq 2.$$

The induction is finished and so $1 \leq a_n \leq 2$ for all n. Next we prove that $\{a_n\}$ is monotone increasing. Note that $a_2 = \frac{3}{2} > a_1 = 1$. Assume that $a_n \geq a_{n-1}$. Then

$$a_{n+1} - a_n = \left(1 + \frac{a_n}{a_n + 1}\right) - \left(1 + \frac{a_{n-1}}{a_{n-1} + 1}\right) = \frac{a_n - a_{n-1}}{(a_n + 1)(a_{n-1} + 1)} \geq 0.$$

Since $\{a_n\}$ is monotone increasing and bounded above, it follows from the Monotone Convergence Theorem that $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = \sup_{n\to\infty} a_n$. Let $A = \lim_{n\to\infty} a_n$. Then

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(1 + \frac{a_n}{a_n + 1}\right) = 1 + \frac{A}{A + 1}$$

and so $A^2 - A - 1 = 0$. It follows that

$$A = \frac{1 \pm \sqrt{5}}{2}.$$

Since $1 \leq a_n \leq 2$, it follows that $1 \leq \lim_{n\to\infty} a_n \leq 2$. Hence, $A = \frac{1-\sqrt{5}}{2}$ is rejected and hence $A = \frac{1+\sqrt{5}}{2}$, that is, $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$. □

3. Let $\{a_n\}$ be a bounded sequence of non-negative real numbers. Show that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \leq 1.$$

Solution. Since $\{a_n\}$ is a bounded sequence of non-negative numbers, there exists a real number M (which may be taken to be positive) such that

$$0 \leq a_n \leq M$$

for all n. Then

$$0 \leq a_n^{\frac{1}{n}} \leq M^{\frac{1}{n}} \quad \forall\, n \geq 1.$$

Thus,

$$\overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} \leq \overline{\lim_{n\to\infty}}\, M^{\frac{1}{n}}.$$

Since the limit $\lim_{n\to\infty} M^{\frac{1}{n}}$ exists, it follows that

$$\overline{\lim_{n\to\infty}}\, M^{\frac{1}{n}} = \lim_{n\to\infty} M^{\frac{1}{n}} = M^0 = 1.$$

Therefore, we have

$$\overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} \leq 1.$$

□

4. If $0 < r < 1$ and $|a_{n+1} - a_n| < r^n$ for all $n \in \mathbb{N}$, prove that $\{a_n\}$ converges.

Solution. For $m > n$, we have

$$\begin{aligned} |a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \\ &\le |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\ &\le r^{m-1} + r^{m-2} + \cdots + r^n = \frac{r^n - r^m}{1-r} \le \frac{r^n + r^m}{1-r}. \end{aligned}$$

Given any $\epsilon > 0$, let N be the smallest integer such that

$$\frac{r^N}{1-r} \le \frac{\epsilon}{2} \quad \text{or} \quad N \ge \frac{\ln \frac{(1-r)\epsilon}{2}}{\ln r}.$$

(**Note.** $\ln r < 0$.) Then, for $m > n > N$,

$$|a_m - a_n| \le \frac{r^n + r^m}{1-r} < \frac{r^N + r^N}{1-r} \le \epsilon.$$

Thus $\{a_n\}$ is a Cauchy sequence and so $\{a_n\}$ converges.

□