NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Take-home Exam 3

1. Determine the convergence or divergence of each of the following series. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+2n-1}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+2\ln n)}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} 2^n \left(1-\frac{1}{n+1}\right)^{n^2}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$.

(v) $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{3}-1)$. [Hint: Try the limit comparison test with the harmonic series. Use $\displaystyle\lim_{n\to\infty}\frac{3^{1/n}-1}{1/n}=\lim_{x\to 0}\frac{3^x-1}{x}$ and then use L'Hospital rule for finding the limit.]

Solution. (i). Since

$$\lim_{k\to\infty}\frac{\dfrac{\sqrt{n}}{n^2+2n-1}}{\dfrac{1}{n^{\frac{3}{2}}}}=\lim_{n\to\infty}\frac{1}{1+2/n-1/n^2}=1,$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by the p-series, the series converges by the limit comparison test.

(ii). Let $f(x)=\dfrac{1}{x(1+2\ln x)}$. Then $f(x)$ is positive and monotone decreasing on $[1,+\infty]$. Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty}\frac{1}{x(1+2\ln x)}dx \xlongequal[\frac{dy}{2}=\frac{dx}{x}]{y=1+2\ln x} \int_1^{\infty}\frac{dy}{2y}=\frac{1}{2}\ln y\Big|_1^{\infty}=+\infty$$

diverges, the series $\sum_{n=1}^{\infty}\frac{1}{n(1+2\ln n)}$ diverges by the integral test.

(iii). Let $a_n = 2^n\left(1-\dfrac{1}{n+1}\right)^{n^2}$. Then

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}2\left(1-\frac{1}{n+1}\right)^n=\lim_{n\to\infty}2\frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{1-\frac{1}{n+1}}=2e^{-1}=\frac{2}{e}<1$$

and so the series is convergent by the root test.

(iv). Let $a_n=\dfrac{2^n\cdot n!}{n^n}$. Since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}\cdot(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot 2^n\cdot n!}$$
$$=\lim_{n\to\infty}\frac{2\cdot(n+1)}{\left(1+\frac{1}{n}\right)^n\cdot(n+1)}=\frac{2}{e}<1,$$

the series is convergent by the ratio test.

(v). Let $a_n = \sqrt[n]{3} - 1$ and let $b_n = \dfrac{1}{n}$. Then

$$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{3^{1/n} - 1}{1/n} \overset{x=\frac{1}{n}}{=\!=\!=} \lim_{x\to 0} \frac{3^x - 1}{x} = \lim_{x\to 0} \frac{3^x \ln 3}{1} = \ln 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{3} - 1)$ diverges. $\square$

2. Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2+1}$, $\quad t \in \mathbb{R}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{(n+2)^n}$.

Solution. (a). Conditional convergence. Let $b_n = \dfrac{1}{2\sqrt{n}+1}$. Then $\{b_n\}$ is positive, monotone decreasing and $\displaystyle\lim_{n\to\infty} b_n = 0$. By the alternating series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+1}$ converges. Since $\dfrac{1}{2\sqrt{n}+1} \geq \dfrac{1}{2\sqrt{n}+\sqrt{n}} = \dfrac{1}{3\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2\sqrt{n}+1} \right| = \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+1}$ diverges.

(b). Absolute convergence. Since

$$\left| \frac{\cos(nt)}{n^2+3} \right| \leq \frac{1}{n^2+3} \leq \frac{1}{n^2}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\cos(nt)}{n^2+3} \right|$ converges by the comparison test.

(c). Divergence. Since

$$\lim_{n\to\infty} \frac{n^n}{(n+2)^n} = \lim_{n\to\infty} \frac{1}{\left(\dfrac{n+2}{n}\right)^n} = \lim_{k\to\infty} \frac{1}{\left(1+\dfrac{2}{n}\right)^n} = \frac{1}{e^2},$$

the limit $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n^n}{(n+2)^n}$ does not exist because it has two subsequential limits $\pm\dfrac{1}{e^2}$. By the divergence test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{(n+2)^n}$ diverges. $\square$