NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 1

Question 1(i). Observe that

$$\left|\frac{2n^2}{3n^2+2}-\frac{2}{3}\right| = \left|\frac{6n^2-2(3n^2+2)}{3(3n^2+2)}\right| = \frac{4}{9n^2+6} \le \frac{4}{9n^2}.$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{4}{9N^2} \le \epsilon$ or $N \ge \frac{2}{3\sqrt{\epsilon}}$. Then

$$\left|\frac{2n^2}{3n^2+2}-\frac{2}{3}\right| \le \frac{4}{9n^2} < \frac{4}{9N^2} \le \epsilon$$

for $n > N$ and hence the result. □

Question 1 (ii). Observe that

$$\left|\sqrt{\frac{n}{n+1}}-1\right| = \left|\frac{\sqrt{n}}{\sqrt{n+1}}-1\right| = \frac{\left|\sqrt{n}-\sqrt{n+1}\right|}{\sqrt{n+1}}$$

$$= \frac{\left|(\sqrt{n}-\sqrt{n+1})(\sqrt{n}+\sqrt{n+1})\right|}{\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})} = \frac{|n-(n+1)|}{\sqrt{n^2+n}+n+1} = \frac{1}{\sqrt{n^2+n}+n+1} \le \frac{1}{n}.$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{1}{N} \le \epsilon$ or $N \ge \frac{1}{\epsilon}$. Then

$$\left|\sqrt{\frac{n}{n+1}}-1\right| \le \frac{1}{n} < \frac{1}{N} \le \epsilon$$

for $n > N$ and hence the result. □

Question 2 (i). This statement is true. We prove the statement by contradiction. Suppose that $\{a_n - b_n\}$ is convergent. Let $c_n = a_n + b_n$. By Theorem 1.4.5, the sequence

$$\{c_n - a_n\} = \{(a_n + b_n) - a_n\} = \{b_n\}$$

is convergent. This contradicts to the assumption that $\{b_n\}$ is divergent and hence $\{a_n + b_n\}$ is divergent. □

Question 2 (ii). Example $a_n = \frac{1}{n}$, $b_n = n$. □

Question 2 (iii). Example: $a_n = n,\ b_n = -n$. □

Question 3 (i).

$$\lim_{n\to\infty}\frac{n^2+5n^3-1}{3n^3+6n+4} = \lim_{n\to\infty}\frac{(n^2+5n^3-1)/n^3}{(3n^3+6n+4)/n^3} = \lim_{n\to\infty}\frac{\frac{1}{n}+5-\frac{1}{n^3}}{3+\frac{6}{n^2}+\frac{4}{n^3}} = \frac{0+5-0}{3+0+0} = \frac{5}{3}.$$

□

Question 3 (ii).

$$\lim_{n\to\infty}\frac{3^n+n^8}{2n^2+7^n}=\lim_{n\to\infty}\frac{(3^n+n^8)/7^n}{(2n^2+7^n)/7^n}=\lim_{n\to\infty}\frac{(\frac{3}{7})^n+\frac{n^8}{7^n}}{\frac{2n^2}{7^n}+1}=\frac{0+0}{0+1}=0.$$

□

Question 3 (iii).

$$\lim_{n\to\infty}\sqrt{\frac{n^4+4n^3+1}{n^3+2n^2}}=\lim_{n\to\infty}\sqrt{\frac{n^4}{n^3}\cdot\frac{(n^4+4n^3+1)/n^4}{(n^3+2n^2)/n^3}}=\lim_{n\to\infty}\sqrt{n}\cdot\sqrt{\frac{1+\frac{4}{n}+\frac{1}{n^4}}{1+\frac{2}{n}}}=+\infty.$$

□

Question 4 (i). Observe that

$$\frac{1}{2n}\le\frac{1+|\sin n|}{2n}\le\frac{2}{2n}=\frac{1}{n}.$$

Since $\lim_{n\to\infty}\frac{1}{2n}=\lim_{n\to\infty}\frac{1}{n}=0$, we have $\lim_{n\to\infty}\frac{1+|\sin n|}{2n}=0$ by the Squeeze Theorem.

□

Question 4 (ii). Since

$$0\le\frac{2n-5}{3n+1}\le\frac{2n}{3n}=\frac{2}{3}$$

for $n\ge 3$, we have

$$0\le\left(\frac{2n-5}{3n+1}\right)^n\le\left(\frac{2}{3}\right)^n$$

for $n\ge 3$. Since $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=\lim_{n\to\infty}0=0$, we have $\lim_{n\to\infty}\left(\frac{2n-5}{3n+1}\right)^n=0$ by the Squeeze Theorem. □

Question 5 (i). Since $\frac{e^n}{n^{100}}>0$ and $\lim_{n\to\infty}\frac{n^{100}}{e^n}=0$, we have $\lim_{n\to\infty}\frac{e^n}{n^{100}}=+\infty$ by the Reciprocal Rule □

Question 5 (ii). Since $\frac{n}{\ln\frac{1}{n+2}}=-\frac{n}{\ln(n+2)}<0$ and

$$\lim_{n\to\infty}-\frac{\ln(n+2)}{n}=\lim_{n\to\infty}-\frac{\frac{1}{n+2}}{1}=0,$$

we have $\lim_{n\to\infty}\frac{n}{\ln\frac{1}{n+2}}=-\infty$ by the Reciprocal Rule. □

Question 6. (a). We will prove (a) by contradiction. Since $\{a_k\}$ converges, we may write $\lim_{k\to\infty}a_k=A\in\mathbb{R}$. Suppose $A<0$. Then choose $\epsilon=-A>0$. Since $\lim_{k\to\infty}a_k=A$, there exists a natural number N such that

$$|a_k-A|<\epsilon=-A\quad\text{for all } k>N. \tag{1}$$

Fix a $k > N$. Then

$$a_k \geq 0 \quad \& \quad A < 0 \implies a_k - A > 0 \implies |a_k - A| = a_k - A.$$

Together with (1), it follows that one has

$$a_k - A < \epsilon = -A \implies a_k < 0,$$

contradicting that $a_k \geq 0$. Thus, we must have $A \geq 0$, i.e., $\lim_{k\to\infty} a_k \geq 0$. □

Question 6. (b). Consider the sequence $\{a_k\}$, where $a_k = c_k - b_k$ for each $k \geq 1$. Since $\{b_k\}$ and $\{c_k\}$ converge, it follows from the subtraction rule that $\{a_k\}$ also converges and

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k - \lim_{k\to\infty} b_k.$$

Since $b_k \leq c_k$, it follows that $a_k = c_k - b_k \geq 0$ for all $k \geq 1$. Hence by (a), we have

$$\lim_{k\to\infty} a_k \geq 0 \implies \lim_{k\to\infty} c_k - \lim_{k\to\infty} b_k \geq 0 \implies \lim_{k\to\infty} c_k \geq \lim_{k\to\infty} b_k.$$

□

Question 6. (c). Consider the constant sequence $\{c_k\}$ where $c_k = L$ for all $k \geq 1$. Then by (ii), we have

$$\lim_{k\to\infty} b_k \leq \lim_{k\to\infty} c_k = \lim_{k\to\infty} L = L.$$

□

Question 6. (d). For any given $\epsilon > 0$, we have $\ln \epsilon \in \mathbb{R}$; since $\lim_{n\to\infty} \ln a_n = -\infty$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$\ln a_n < \ln \epsilon \quad \text{for all } n > N$$
$$\implies a_n < \epsilon \quad \text{for all } n > N \quad (\text{since } f(x) = \ln x \ \text{ increases with } x)$$
$$\implies |a_n - 0| < \epsilon \quad \text{for all } n > N \quad (\text{since } a_n > 0).$$

Thus, $\lim_{n\to\infty} a_n = 0$. □