NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 10

Question 1 (i).

$$R=\frac{1}{\lim\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{|-2|^{n+1}\cdot n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}\cdot|-2|^n}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{2}{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}}=\frac{1}{2}.$$

Consider the ending points $x=x_0\pm R=\pm\dfrac{1}{2}$. The series $\displaystyle\sum_{n=1}^{\infty}\frac{[-2\cdot(-\frac{1}{2})]^n}{n^{\frac{3}{2}}}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is convergent by the p-series and the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-2\cdot\frac{1}{2})^n}{n^{\frac{3}{2}}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{3}{2}}}$ is convergent by the alternating series test. Thus the interval of convergence is $[-\dfrac{1}{2},\dfrac{1}{2}]$. □

Question 1 (ii).

$$R=\frac{1}{\lim\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{3^{n+1}\cdot(n+1)}{(n+2)\cdot 3^n}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{3\cdot\left(1+\frac{1}{n}\right)}{1+\frac{2}{n}}}=\frac{1}{3}.$$

Consider the ending points $x=x_0\pm R=2\pm\dfrac{1}{3}$. When $x=2-\dfrac{1}{3}$,

$$\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n+1}=\sum_{n=1}^{\infty}\frac{3^n\left(-\frac{1}{3}\right)^n}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

is convergent by the alternating series test. When $x=2+\dfrac{1}{3}$,

$$\sum_{n=1}^{\infty}\frac{3^n(x-2)^n}{n+1}=\sum_{n=1}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{n+1}=\sum_{n=1}^{\infty}\frac{1}{n+1}$$

is divergent by the p-series. Thus the interval of convergence is $\left[2-\dfrac{1}{3},2+\dfrac{1}{3}\right)$. □

Question 1 (iii). Observe that

$$\sum_{n=1}^{\infty}\frac{(1-3x)^n}{n}=\sum_{n=1}^{\infty}\frac{(-3)^n}{n}\cdot\left(x-\frac{1}{3}\right)^n.$$

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{|-3|^n}{n}}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3}{\sqrt[n]{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = \frac{1}{3} \pm \frac{1}{3}$. When $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent by the p-series. When $x = \frac{2}{3}$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent by the alternating series test. Thus the interval of convergence is $\left(0, \frac{2}{3}\right]$. □

Question 2(a). $f'(x) = 2e^{2x}$, $f''(x) = 2^2 e^{2x}$ and in general $f^{(k)}(x) = 2^k e^{2x}$. Thus

$$e^{2x} \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{2^k e^6}{k!}(x-3)^k = e^6 \sum_{k=0}^{\infty} \frac{2^k}{k!}(x-3)^k.$$

$$T_3(f,4) = e^8 + e^8 2(x-4) + e^8 2(x-4)^2 + \frac{e^8 4}{3}(x-4)^3.$$

□

Question 2(b). (i)

$$\begin{aligned} \sin^2(4x) &= \frac{1-\cos 8x}{2} \\ &= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n}}{(2n)!}, \quad |8x| < \infty \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n 8^{2n}}{2\cdot(2n)!} x^{2n}, \quad |x| < \infty \end{aligned}$$

(ii)

$$\begin{aligned} \frac{1}{(x+1)(2x+1)} &= \frac{-1}{x+1} + \frac{2}{2x+1} \\ &= \frac{-1}{(x-1)+2} + \frac{2}{2(x-1)+3} \\ &= -\frac{1}{2}\cdot\frac{1}{1+\frac{x-1}{2}} + \frac{2}{3}\cdot\frac{1}{1+\frac{2(x-1)}{3}} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{2}\right)^n + \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2(x-1)}{3}\right)^n \end{aligned}$$

$$= \sum_{n=0}^{\infty}[\frac{(-1)^{n+1}}{2^{n+1}} + \frac{(-1)^n 2^{n+1}}{3^{n+1}}](x-1)^n.$$

☐

Question 3. (i). We have

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n, \quad |t| < 1.$$

In particular, the radius of convergence r of the above power series is at least 1 (one may also directly verify that $r = 1$). Thus, for each fixed x with $|x| < 1$, the above power series converges uniformly to $f(t) = \dfrac{1}{1+t}$ on the interval $|t| \leq |x|$ (and thus also on the smaller interval $[0, x]$. Also, each function $f_n(t) = (-1)^n t^n$ is continuous. Thus, we have

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n dt$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n},$$

where

$$\int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n dt$$

because the power series $\sum\limits_{n=0}^{\infty}(-1)^n t^n$ converges at $t = 0, x$.

(ii).

$$\ln(1+2x^2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(2x^2\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 2^n}{n}x^{2n}$$

for $2x^2 < 1$ or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and so the Maclaurin series of $\ln(1+2x^2)$ is $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 2^n}{n}x^{2n}$.

☐

Question 4. Let $f(x) = \cos x$. Then

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

By Taylor's Theorem, we have

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

Observe that

$$0 \le |R_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}.$$

By the Standard Limits, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. Thus $\lim_{n\to\infty} |R_n(x)| = 0$ by the Squeeze Theorem and so $\lim_{n\to\infty} R_n(x) = 0$. Hence

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \lim_{n\to\infty} f(x) - R_n(x) = f(x) - 0 = f(x).$$

It follows that

$$\cos x = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}.$$

□

Question 5. By standard power series, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty).$$

Replace x by x^2, we have

$$\begin{aligned} \sin(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} \quad (|x^2| < \infty) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \quad (|x| < \infty). \end{aligned}$$

The series of functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$ has radius of convergence $R = +\infty$ (why?), and thus it converges uniformly on $[0, 0.2]$. Moreover, each function $\frac{(-1)^n x^{4n+2}}{(2n+1)!}$ is continuous on $[0, 0.2]$. Thus,

$$\begin{aligned} \int_0^{0.2} \sin x^2 \, dx &= \int_0^{0.2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \, dx \\ &= \sum_{n=0}^{\infty} \int_0^{0.2} (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(0.2)^{4n+3}}{(2n+1)! \cdot 4n+3} \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)! \cdot (4n+3) \cdot 5^{4n+3}}.$$

Let $a_n = \dfrac{1}{(2n+1)! \cdot (4n+3) \cdot 5^{4n+5}}$. Then a_n is positive, monotone decreasing and $\lim\limits_{n\to\infty} a_n = 0$. Thus we may apply the alternating series estimation.
When $n = 0$, error $\leq a_1 = \frac{1}{3! \cdot 7 \cdot 5^7} \leq 3 \times 10^{-6}$.
When $n = 1$, error $\leq a_2 = \frac{1}{5! \cdot 11 \cdot 5^{11}} \leq 1.6 \times 10^{-11} < 10^{-8}$. Thus,

$$\int_0^{0.2} \sin x^2 dx \approx \frac{1}{3 \cdot 5^3} - \frac{1}{3! \cdot 7 \cdot 5^7} = \frac{1}{375} - \frac{1}{3281250} \approx 0.00266636$$

with error $< 10^{-8}$.

$\square$