NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 2

Question 1 (1).

$$\lim_{n\to\infty}\sqrt[3]{\frac{2n^4+n+1}{16n^4+n^2+2}}=\lim_{n\to\infty}\sqrt[3]{\frac{(2n^4+n+1)/n^4}{(16n^4+n^2+2)/n^4}}$$

$$=\lim_{n\to\infty}\sqrt[3]{\frac{2+\frac{1}{n^3}+\frac{1}{n^4}}{16+\frac{1}{n^2}+\frac{2}{n^4}}}=\sqrt[3]{\frac{2+0+0}{16+0+0}}=\sqrt[3]{\frac{1}{8}}=\frac{1}{2}.$$

□

Question 1 (2).

$$\lim_{n\to\infty}\left(3+\ln\left(\cos\frac{1}{\sqrt{n}}\right)+\frac{n^2}{1.1^n}\right)=3+\ln(\cos 0)+0=3.$$

□

Question 1 (3).

$$\lim_{n\to\infty}\frac{n^4+8^n}{9^n+n+8^n}=\lim_{n\to\infty}\frac{(n^4+8^n)/9^n}{(9^n+n+8^n)/9^n}=\lim_{n\to\infty}\frac{\frac{n^4}{9^n}+\left(\frac{8}{9}\right)^n}{1+\frac{n}{9^n}+\left(\frac{8}{9}\right)^n}=\frac{0+0}{1+0+0}=0.$$

□

Question 1 (4).

$$\lim_{n\to\infty}\sqrt[n]{\frac{n}{3}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{3^{\frac{1}{n}}}=\frac{1}{3^0}=1.$$

□

Question 1 (5).

$$\lim_{n\to\infty}\left(\sqrt{3n-2}-\sqrt{3n-3}\right)=\lim_{n\to\infty}\frac{\left(\sqrt{3n-2}-\sqrt{3n-3}\right)\left(\sqrt{3n-2}+\sqrt{3n-3}\right)}{\sqrt{3n-2}+\sqrt{3n-3}}$$

$$=\lim_{n\to\infty}\frac{(3n-2)-(3n-3)}{\sqrt{3n-2}+\sqrt{3n-3}}=\lim_{n\to\infty}\frac{1}{\sqrt{3n-2}+\sqrt{3n-3}}=0.$$

□

Question 1 (6). Since $0\le\frac{3+(-1)^n}{5}\le\frac{4}{5}$, we have $0\le\left(\frac{3+(-1)^n}{5}\right)^n\le\left(\frac{4}{5}\right)^n$. Since $\lim_{n\to\infty}\left(\frac{4}{5}\right)^n=\lim_{n\to\infty}0=0$, we have $\lim_{n\to\infty}\left(\frac{3+(-1)^n}{5}\right)^n=0$ by the Squeeze Theorem. □

Question 1 (7).

$$\lim_{n\to\infty}\frac{7^n+\ln n-n!}{n!+n^2}=\lim_{n\to\infty}\frac{(7^n+\ln n-n!)/n!}{(n!+n^2)/n!}=\lim_{n\to\infty}\frac{\frac{7^n}{n!}+\frac{\ln n}{n!}-1}{1+\frac{n^2}{n!}}=\frac{0+0-1}{1+0}=-1.$$

□

Question 1 (8).

$$\lim_{n\to\infty}\frac{n^{100}100^n}{n!}=\lim_{n\to\infty}\frac{n^{100}}{2^n}\cdot\frac{2^n\cdot 100^n}{n!}=\lim_{n\to\infty}\frac{n^{100}}{2^n}\cdot\frac{200^n}{n!}=0\cdot 0=0.$$

□

Question 1 (9). Observe that

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}}=\lim_{n\to\infty}\frac{2}{\sqrt{n}}=0.$$

Since

$$-\frac{\ln n}{\sqrt{n}}\le\frac{(-1)^{n+1}\ln n}{\sqrt{n}}\le\frac{\ln n}{\sqrt{n}},$$

we have $\lim_{n\to\infty}\frac{(-1)^{n+1}\ln n}{\sqrt{n}}=0$ by the Squeeze Theorem. □

Question 1 (10).

$$\lim_{n\to\infty}(3^n+4^n)^{\frac{1}{n}}=\lim_{n\to\infty}(4^n)^{\frac{1}{n}}\cdot\left(\left(\frac{3}{4}\right)^n+1\right)^{\frac{1}{n}}=\lim_{n\to\infty}4\left(\left(\frac{3}{4}\right)^n+1\right)^{\frac{1}{n}}=4\cdot(0+1)^0=4.$$

□

Question 1 (11).

$$\lim_{n\to\infty}n\sin\frac{3}{n}=\lim_{n\to\infty}3\cdot\frac{\sin\frac{3}{n}}{\frac{3}{n}}=\lim_{x\to 0}3\cdot\frac{\sin x}{x}=3\cdot 1=3,$$

where $x=\dfrac{3}{n}$. □

Question 1 (12).

$$\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{2n}=\lim_{n\to\infty}\left(\left(1+\frac{-\frac{1}{3}}{n}\right)^n\right)^2=\left(e^{-\frac{1}{3}}\right)^2=e^{-\frac{2}{3}}.$$

□

Question 6 (13). $\lim_{n\to\infty}(n^2+1)^{\frac{1}{\ln(n+1)}}$. Let $a_n=(n^2+1)^{\frac{1}{\ln(n+1)}}$. Then

$$\lim_{n\to\infty}\ln a_n=\lim_{n\to\infty}\frac{\ln(n^2+1)}{\ln(n+1)}=2\quad\text{by L'Hopital's rule}$$

$$\Longrightarrow \ln\big(\lim_{n\to\infty}\ln a_n\big)=\lim_{n\to\infty}\ln a_n=2$$

$$\Longrightarrow \lim_{n\to\infty} a_n = e^2.$$

□

Question 2. Since S, T are bounded, there exist $m_1, m_2, M_1, M_2 \in \mathbb{R}$ such that

$$m_1 \le x \le M_1 \quad \text{for all } x \in S, \qquad \text{and} \qquad m_2 \le x \le M_2 \quad \text{for all } x \in T.$$

Then for all $z \in S \cup T$, we have

$$z \in S \text{ or } z \in T$$
$$\Longrightarrow m_1 \le z \le M_1 \text{ or } m_2 \le z \le M_2$$
$$\Longrightarrow \min\{m_1, m_2\} \le z \le \max\{M_1, M_2\}.$$

Therefore, $S \cup T$ is bounded (above by $M = \max\{M_1, M_2\}$ and below by $\min\{m_1, m_2\}$).

□

Question 3 (i). Since $\{a_n\}$ is bounded, there exists $m, M \in \mathbb{R}$ such that $m \le a_n \le M$ (and thus $-m \ge -b_n \ge -M$) for all n.

$$|a_n| = \begin{cases} a_n & \text{if} \quad a_n \ge 0, \\ -a_n & \text{if} \quad a_n < 0, \end{cases} \le \begin{cases} M & \text{if} \quad a_n \ge 0, \\ -m & \text{if} \quad a_n < 0 \end{cases} \le \max\{-m, M\}.$$

Therefore, $\{|a_n|\}$ is bounded above (by $\max\{-m, M\}$).

Clearly, $|a_n| \ge 0$ for all $n \ge 1$. $\{|a_n|\}$ is bounded below (by $m = 0$).
Since $\{|a_n|\}$ is bounded above and below, $\{|a_n|\}$ is a bounded sequence.

□

Question 3 (ii). Since $\{b_n\}$ is bounded, the sequence $\{|b_n|\}$ is bounded by (i) and so there exists $M \in \mathbb{R}$ such that $|b_n| \le M$ for all n. Clearly, $M \ge 0$, since each $|b_n| \ge 0$. Without loss of generality, we may assume that $M > 0$, since if M is an upper bound of $\{|b_n|\}$, so is any number $M' > M$.

Given any $\epsilon > 0$, since $\lim_{n\to\infty} a_n = 0$, there exists a positive integer N such that $|a_n| = |a_n - 0| < \dfrac{\epsilon}{M}$ for all $n > N$. It follows that

$$|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| \le |a_n| \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $n > N$ and so $\lim_{n\to\infty} a_n b_n = 0$. □

Question 3 (iii). Example: $c_n = \dfrac{1}{n}, \quad d_n = n.$ □