

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

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MA2108

Advanced Calculus II

Solutions to Tutorial 3

Question 1. Answer: $\sup S = 1$ and $\inf S = 0$. We show that $\sup S = 1$. Let r be any element in S . By the definition, r is rational number with $0 \leq r < 1$. Thus 1 is an upper bound of S and 0 is a lower bound of S . Let M be any upper bound of S . Then $r \leq M$ for any rational number r with $0 \leq r < 1$. In particular, $\frac{n}{n+1} \leq M$ for any positive integer n . It follows that $1 = \lim_{n \rightarrow \infty} \frac{n}{n+1} \leq M$ and so 1 is the least upper bound of S or $\sup S = 1$. Let m be any lower bound of S . Then $m \leq r$ for any rational number r with $0 \leq r < 1$. In particular, $m \leq 0$ and so 0 is the greatest lower bound of S or $\inf S = 0$. \square

Question 2 (i). First we show that $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. Let z be any element in $A \cup B$. Then $z \in A$ or B . If $z \in A$, then $z \leq \sup A \leq \max\{\sup A, \sup B\}$. Otherwise, $z \in B$ and $z \leq \sup B \leq \max\{\sup A, \sup B\}$. Thus $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$.

Now we show that $\max\{\sup A, \sup B\}$ is the least upper bound of $A \cup B$. Let M be any upper bound of $A \cup B$. Then $z \leq M$ for any $z \in A \cup B$. In particular, $z \leq M$ for $z \in A \subseteq A \cup B$ and so M is an upper bound of A . It follows that $\sup A \leq M$. Similarly, we have $\sup B \leq M$. Thus $\max\{\sup A, \sup B\} \leq M$ and so $\max\{\sup A, \sup B\} = \sup A \cup B$. \square

Question 2 (ii). No, it is not true. A counter-example is as follows. Let $A = \{1, 2\}$ and let $B = \{1, 3\}$. Then $\sup A = \max A = 2$ and $\sup B = \max B = 3$. It follows that $\min\{\sup A, \sup B\} = \min\{2, 3\} = 2$. But $\sup A \cap B = \sup\{1\} = 1 \neq \min\{\sup A, \sup B\}$. \square

Question 3 (i). Let $M = \sup A + \sup B$. We are going to show that M satisfies (S-i) and (S-ii) for C .

(S-i): For any $x \in C$,

$$\begin{aligned} x &= a + b \quad \text{for some } a \in A, b \in B \\ \implies x &\leq \sup A + \sup B, \text{ since } a \leq \sup A, b \leq \sup B \\ &\implies x \leq M. \end{aligned}$$

Thus, M is an upper bound of C .

(S-ii): Suppose M' is an upper bound of C . Then

$$\begin{aligned}
& M' \geq x \quad \text{for all } x \in C \\
\implies & M' \geq a + b \quad \text{for all } a \in A \ \& \ b \in B \quad (\text{since such } a + b \in C) \\
& \implies M' - b \geq a \quad \text{for all } a \in A \ \& \ b \in B \\
\implies & M' - b \geq \sup A \quad \text{for all } b \in B \quad (\text{by (S-ii) for } A) \\
& \implies M' - \sup A \geq b \quad \text{for all } b \in B \\
\implies & M' - \sup A \geq \sup B \quad (\text{by (S-ii) for } B) \\
& \implies M' \geq \sup A + \sup B = M.
\end{aligned}$$

Thus, $M = \sup A + \sup B$ satisfies (S-ii) for C .

Since $M = \sup A + \sup B$ satisfies (S-i) and (S-ii) for C , it follows that

$$\sup C = \sup A + \sup B.$$

□

Question 3(ii). Let $A = B = \{0, 1\}$. Then $\sup A = \sup B = 1$ and thus

$$\sup A - \sup B = 1 - 1 = 0.$$

Now, $D = \{0 - 0, 0 - 1, 1 - 0, 1 - 1\} = \{0, -1, 1\}$. Thus,

$$\sup D = 1 \neq 0 = \sup A - \sup B.$$

□

Question 4 (i). We prove that $2 \leq a_n \leq 3$ by induction on n . Since $a_1 = 2$, we have $2 \leq a_1 \leq 3$. Suppose that $a_{n-1} \leq 3$ with $n \geq 2$. Then

$$2 \leq \sqrt{6 + 2} \leq a_n = \sqrt{6 + a_{n-1}} \leq \sqrt{6 + 3} = 3.$$

The induction is finished and hence the statement.

□

Question 4 (ii). Let $n \geq 2$. Then

$$\begin{aligned}
a_n - a_{n-1} &= \sqrt{6 + a_{n-1}} - a_{n-1} = \frac{(\sqrt{6 + a_{n-1}} - a_{n-1})(\sqrt{6 + a_{n-1}} + a_{n-1})}{\sqrt{6 + a_{n-1}} + a_{n-1}} \\
&= \frac{6 + a_{n-1} - a_{n-1}^2}{\sqrt{6 + a_{n-1}} + a_{n-1}} \geq 0
\end{aligned}$$

because $\sqrt{6 + a_{n-1}} + a_{n-1} > 0$ and $6 + x - x^2 = -(x - 3)(x + 2) \geq 0$ for $-2 \leq x \leq 3$. Thus $\{a_n\}$ is monotone increasing. □

Question 4 (iii). By (i) and (ii), $\{a_n\}$ is bounded above and monotone increasing. Thus $\{a_n\}$ is convergent. Let $A = \lim_{n \rightarrow \infty} a_n$. Then we have the equation

$$A = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{6 + a_{n-1}} = \sqrt{6 + \lim_{n \rightarrow \infty} a_{n-1}} = \sqrt{6 + A}$$

and so $A^2 = 6 + A$. It follows that $A = -2$ or 3 . Since $a_n \geq 2$ for each n , $A = \lim_{n \rightarrow \infty} a_n \geq 2$ and so $A = 3$. □

Question 5. A simple check using calculator gives

$$x_1 = 0.75, x_2 = 0.938, x_3 = 0.996, x_4 = 0.999, \dots,$$

which leads to the guess that $\{x_n\}$ is bounded above by 1.

First we show that $0 \leq x_n \leq 1$ by induction on n . When $n = 1$, we have $0 \leq x_1 = \frac{3}{4} \leq 1$. Suppose that $0 \leq x_{n-1} \leq 1$ with $n \geq 2$. Observe that

$$x_n = 2x_{n-1} - x_{n-1}^2 = 1 - (1 - 2x_{n-1} + x_{n-1}^2) = 1 - (1 - x_{n-1})^2.$$

Since $0 \leq 1 - x_{n-1} \leq 1$ by induction, we have $0 \leq x_n \leq 1$. The induction is finished and so $0 \leq x_n \leq 1$ for all n .

Observe that

$$x_{n+1} - x_n = (2x_n - x_n^2) - x_n = x_n - x_n^2 = x_n(1 - x_n).$$

Since $0 \leq x_n \leq 1$, we have $x_{n+1} - x_n = x_n(1 - x_n) \geq 0$ and so the sequence $\{x_n\}$ is monotone increasing and bounded. Thus the limit of $\{x_n\}$ exists. Let $A = \lim_{n \rightarrow \infty} x_n$.

From the equation $x_{n+1} = 2x_n - x_n^2$, we have

$$A = \lim_{n \rightarrow \infty} x_{n+1} = 2 \lim_{n \rightarrow \infty} x_n - \left(\lim_{n \rightarrow \infty} x_n \right)^2 = 2A - A^2.$$

and so $A = 0$ or 1 . Since $x_n \geq x_1 = \frac{3}{4}$, we have $A = \lim_{n \rightarrow \infty} x_n \geq \frac{3}{4}$ and so $A \neq 0$. Thus $A = 1$. □