NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 4

Question 1 (i). $\{a_n\} = \{1, 0, 1, 0, 1, 0, \cdots\}$. Thus,

$$\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} 0 = 0, \quad \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} 1 = 1.$$

Thus, $\{a_n\}$ has two subsequences which converge to two different limits. Then $\{a_n\}$ diverges. $\square$

Question 1(ii). $\{a_n\} = \{1, 1, 1, \cdots\}$. Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1$(converges).

$\square$

Question 1(iii). $\{a_n\}$ diverges (to $+\infty$). Apply the reciprocal rule. $\square$

Question 2. Given any $\epsilon > 0$, since $\lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} a_{2k} = A$, there exists $K_1, K_2 \in \mathbb{N}$ such that

$$|a_{2k-1} - A| < \epsilon \quad \forall k > K_1, \text{ and} \tag{1}$$

$$|a_{2k} - A| < \epsilon \quad \forall k > K_2. \tag{2}$$

Choose $N = \max(2K_1 - 1, 2K_2)$. Then for all $n > N$,
Case (a): n is odd. In this case, $n = 2k - 1$. Then $n > N \implies 2k - 1 > 2k_1 - 1 \implies k > K_1$. Hence we have $|a_n - A| = |a_{2k-1} - A| < \epsilon$ by (1).
Case (b): n is even. In this case, $n = 2k$. Then $n > N \implies 2k > 2k_2 \implies k > K_2$. Thus we have $|a_n - A| = |a_{2k} - A| < \epsilon$ by (2).
Hence we have $|a_n - A| < \epsilon$ for all $n > N$. Therefore, $\lim_{n\to\infty} a_n = A$.

$\square$

Question 3 (a). Since $\{a_n\} = \{4 + \cos\frac{n\pi}{2}\} = \{4, 3, 4, 5, 4, 3, 4, 5, 4, 3, 4, 5, \cdots\}$, we have

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 5 \quad \text{and} \quad c_n = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 3$$

for all n and so $\overline{\lim_{n\to\infty}}\, 4 + \cos\frac{n\pi}{2} = \lim_{n\to\infty} b_n = 5$ and $\underline{\lim_{n\to\infty}}\, 4 + \cos\frac{n\pi}{2} = \lim_{n\to\infty} c_n = 3$. $\square$

Question 3 (b). Observe that

$$0 \le \frac{1 + (-1)^n}{n} \le \frac{2}{n}.$$

Since $\lim_{n\to\infty} \frac{2}{n} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} \frac{1 + (-1)^n}{n} = 0$ and so

$$\overline{\lim_{n\to\infty}} \frac{1 + (-1)^n}{n} = \underline{\lim_{n\to\infty}} \frac{1 + (-1)^n}{n} = \lim_{n\to\infty} \frac{1 + (-1)^n}{n} = 0.$$

$\square$

Question 4(i). Let $e_n = \inf_{k\geq n}(a_k+b_k)$, $c_n = \inf_{k\geq n} a_k$, $d_n = \inf_{k\geq n} b_k$. Then for any fixed n and any $k \geq n$, we have

$$a_k + b_k \geq d_n + e_n \quad (\text{since } a_k \geq d_n,\ b_k \geq e_n)$$
$$\Longrightarrow d_n + e_n \text{ is a lower bound of } \{a_n + b_n, a_{n+1} + b_{n+1}, \cdots\}$$
$$\Longrightarrow d_n + e_n \leq \inf_{k\geq n}(a_k + b_k) = c_n.$$

Thus, $\liminf_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} c_n \geq \lim_{n\to\infty} d_n + \lim_{n\to\infty} e_n = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$. □

Question 4(ii). $\{a_n\} = \{0, 1, 0, 1, 0, 1, \cdots\}$, $\{b_n\} = \{1, 0, 1, 0, 1, 0, 1, 0, \cdots\}$. Then $\{a_n + b_n\} = \{1, 1, 1, \cdots\}$. Now, $\liminf a_n = \liminf b_n = 0$. Thus,

$$\liminf(a_n + b_n) = 1 > 0 = 0 + 0 = \liminf a_n + \liminf b_n = 0.$$

□

Question 5. Since $\{a_n\}$ is Cauchy sequences, it follows from Cauchy's criterion that $\{a_n\}$ is convergent. By the product rule and the addition rule, it follows that that $\{a_n^2 + a_n\} = \{a_n \cdot a_n + a_n\}$ is also convergent. By Cauchy's criterion again, it follows that $\{a_n^2 + a_n\}$ is also a Cauchy sequence. □